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With an Introduction by NN

1 Introduction and Historical Remarks

This section will be written by colleagues and friends, who work in this field much longer as I do. They will relate our work to the literature and give references.

2 Classical Theory

The traditional way of looking at preferences is: Measure the value or utility of elements $a, b \in A$. If the value of a is "higher" than that of b, then prefer a to b ($a \prec b$). If it is the same, don't care ($a \sim b$). Very often IR, the set of real numbers is taken to measure values. But there are many other meaningful value-sets of utilities like: {bad, medium good}, $\{\text{roughly } r \mid r \in IR\}, [0,1]$ etc.

In general we just need arbitrary linearly ordered sets I to serve the purpose. In the next step, one considers several different measures (or different persons doing measurements), several criteria or several future scenarios. This leads to $I_1 \times I_2 \times \ldots \times I_k$ as the natural utility. This set is no longer a linear order, it is "only" a partial order. Using it to compare elements $a, b \in A$ may lead to pairs that are not comparable, i.e. we may not find a "best" one. Much of the literature is devoted to the question how to make it "deciding" again. The standard technique is to study functions (aggregations):

$$ag : I_1 \times \ldots \times I_k \to I,$$

where I is also some linear order. Especially $ag : [0,1]^k \to [0,1]$ is studied intensively in recent work [ ]. The most general view would be to replace $I_1 \times \ldots I_k$ by some arbitrary partial order V and study aggregations $ag : V \to I$. Note that "aggregations" then are nothing but order preserving mappings.

3 Preference Structures

In this section, we start our considerations from zero, i.e. without any prerequisites as discussed briefly earlier.

We study the following (abstract) situation:

$A = \{a_1, \dots, a_n\}$ a set of alternatives (choices, $\dots$).

Question : Do you prefer a_i to a_j?

Possible Answers:

1. I prefer a_i to a_j : $a_i \prec a_j$
2. I prefer a_j to a_i : $a_j \prec a_i$
3. They look similar to me : $a_i \sim a_j$
4. I don't know (yet) or
   I don't care : $a_i \lozenge a_j$

Of course there are other possible answers like "very much", "to some extend (degree)", "possibly", "with a certain probability", "with high probability", or even "if you do so".

In this paper we will not deal with these more complicated cases, they may be treated later.
From the intuition we have about our four possible answers to Q, we deduce the following definition:

Definition 3.1
$\mathrm{A} = (A; \prec, \sim, \lozenge)$ is a weak preference structure :$\Leftrightarrow$ $\prec, \sim, \lozenge$ are binary relations on A such that

1. $a \sim a$
2. $a \sim b \Rightarrow b \sim a$
3. $a \lozenge b \Rightarrow b \lozenge a$
4. $a \prec b$ or $b \prec a$ or $a \sim b$ or $a \lozenge b$

Relations $\sim$ that fulfill 1. and 2. are called "similarities". Note that $a \prec b$ and $b \prec a$ (as an example) are possible for a weak preference structure. This is needed if we want to model "some people prefer a to b" or in situations where the intuition of some people is to be modeled as f.i. in [ ]. In what we want to study here, these "conflicts" should be excluded.

Definition 3.2
(a, b) is a conflict of the weak preference structure $\mathrm{A} = (A; \prec, \sim, \lozenge)$
:$\Leftrightarrow$ $(a \prec b$ and $b \prec a)$ or $(a \prec b$ and $a \sim b)$ or $(a \prec b$ and $a \lozenge b)$ or $(a \sim b$ and $a \lozenge b)$

The proof of the following lemma is obvious.

Lemma 3.3
$\mathrm{A} = (A; \prec, \sim, \Diamond)$ is a conflict free weak preference structure iff

1. $a \sim a$
2. $a \sim b \Rightarrow b \sim a$
3. $a \prec b \Rightarrow \text{not}(b \prec a)$
4. $a \prec b \Rightarrow \text{not } (a \sim b)$
5. $a \Diamond b \Leftrightarrow \text{not } (a \prec b \text{ or } b \prec a \text{ or } a \sim b)$

Lemma 3.3 leads us to the following final definition of preference structures.

Definition 3.4
$\mathrm{A} = (A; \prec, \sim)$ is a <u>preference structure</u> $:\Leftrightarrow$ A is a finite set, $\prec, \sim$ binary relations on A such that

1. $a \sim a$
2. $a \sim b \Rightarrow b \sim a$
3. $a \prec b \Rightarrow \text{not}(b \prec a)$
4. $a \prec b \Rightarrow \text{not } (b \sim a)$

As a short notation we keep $a \Diamond b$ for not($a \prec b$ or $b \prec a$ or $a \sim b$). Note that 3 in definition 3.4 is equivalent to the more common

$$a \leq b \text{ and } b \leq a \Rightarrow a = b$$

where $\leq$ has the obvious meaning $<$ or $=$. It may be interesting, that this can be extended in the following way.

Lemma 3.5
Let $\mathrm{A} = (A; \prec, \sim)$ be a preference structure and $a \precsim b$ if $a \prec b$ or $a \sim b$. Then

$$a \precsim b \text{ and } b \precsim a \Rightarrow a \sim b$$

Proof. If not ($a \sim b$), then $a \prec b$ and $b \prec a$ which is excluded by definition.□

We are able to define interesting subclasses of preference structures even on this very general level. For the rest of this paper, $\mathrm{A} = (A; \prec, \sim)$ always denotes a preference structure.

Definition 3.6
$\mathrm{A} = (A; \prec, \sim)$ is <u>deciding</u> $:\Leftrightarrow$ There exists an $a \in A$ such that $b \prec a$ or $a \sim b$ for all $b \in A$. a is called strongly maximal.

Definition 3.7
$\mathrm{A} = (A; \prec, \sim)$ is <u>total</u> $:\Leftrightarrow$ not ($a \Diamond b$) for all $a, b \in A$.

Definition 3.8

$A = (A;\prec,\sim)$ is <u>crisp</u> $:\Leftrightarrow$ ($a \sim b$, $b\sim c \Rightarrow a\sim c$) i.e. $\sim$ is an equivalence relation.

Note that $A = (A;\prec,\sim)$ may be total and not deciding, f.i. if

$$a_1 \prec a_2 \prec a_3 \prec \ldots \prec a_k \prec a_1.$$

But if A is total and has a maximal element, then A is deciding. To exclude cycles like the one above, it is sufficient to have transitivity on $\prec$. But the following simple example shows, that preference structures may contain cycles, but still are deciding.

Example 3.9

$$A = \{a_1, a_2, a_3, a_4\}.$$
$$a_1 \prec a_2, a_2 \prec a_3, a_3 \prec a_1, a_i \prec a_4, i \in \{1,2,3\}.$$

Clearly, a_4 is a maximal element.

4 Possible Axioms for Rationality

We start our discussion with two hot candidates for axioms.

$$T_1: \; a \prec b, b \prec c \Rightarrow a \prec c$$
$$A_1: \; a \prec b, a' \sim a, b' \sim b \Rightarrow a' \prec b'$$

T_1 brakes cycles and A_1 seems so natural that we would like to consider it. But the following variation of T_1 may be "natural" also in the light of A_1:

$$a \prec b, c \prec d \text{ and } b \sim c \Rightarrow a \prec d$$

At the other hand consider the following example :

Example 4.1

$A = \{1, 2, 3, 4\}$
$a \sim b \Leftrightarrow | a - b | \leq 2$
$a \prec b \Leftrightarrow a \leq b$ and not ($a \sim b$)
i.e.
$1 \prec 4$, $1 \sim 2$, $3 \sim 4$ <u>and</u> $2 \sim 3$
(but not $2 \prec 3$ as from A_1)

This seems to be a very fundamental example: If we can "measure" the elements of A we take $a \sim b$ if they are "close enough". This example violates A_1 but not

$$A_2: \; a \prec b, a' \sim a, b' \sim b \Rightarrow a' \prec b' \text{ or } a' \sim b'$$

Our conclusion is, that we should start from as many variations of A_1 and T_1 and find reasons to reduce the number of pairs as far as possible. Let's start with the T's.

Table 4.2

$T_1: \; a \prec b, b \prec c \Rightarrow a \prec c$
$T_2: \; a \prec b, b \prec c \Rightarrow a \prec c \text{ or } a \sim c$
$T_3: \; a \prec b, b \prec c \Rightarrow \text{not } (c \prec a)$
$T_4: \; a \prec b, c \prec d, b \sim c \Rightarrow a \prec d$
$T_5: \; a \prec b, c \prec d, b \sim c \Rightarrow a \prec d \text{ or } a \sim d$
$T_6: \; a \prec b, c \prec d, b \sim c \Rightarrow \text{not } (d \prec d)$
$T_7: \; a \prec b, b \prec c \Rightarrow a' \prec c' \text{ for some } a \sim a', c \sim c'$
$T_8: \; a \prec b, c \prec d, b \sim c \Rightarrow a' \prec d' \text{ for some } a' \sim a, d' \sim d$

There are some obvious implications, which are summarized in the following diagram:

Diagram 4.3

$$\begin{array}{ccccccc} T_8 & \Leftarrow & T_4 & \Rightarrow & T_5 & \Rightarrow & T_6 \\ \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\ T_7 & \Leftarrow & T_1 & \Rightarrow & T_2 & \Rightarrow & T_3 \end{array}$$

Before we look at variations of A_1, let's prove the following:

Lemma 4.4
If T_1 holds, then

$$A_6: \; a \prec b, a' \sim a, b' \sim b \Rightarrow \text{not } (b \prec a') \text{ and not } (b' \prec a).$$

Proof. Assume, the conclusion in A_6 does not hold. Then $b \prec a'$ or $b' \prec a$. By T_1 we conclude from $a \prec b$ that $b' \prec b$ or $a' \prec a$, both of which cannot be true by the definition of preference structures. □

A_6 may be considered a very weak variation of A_1. More variations are given in the following:

Table 4.5

$$
\begin{array}{l}
\quad a \prec b,\ a' \sim a,\ b' \sim b \Rightarrow \\
A_1 : a' \prec b' \\
A_2 : a' \prec b' \text{ or } a' \sim b' \\
A_3 : (a' \prec b \text{ or } a' \sim b) \text{ and } (a \prec b' \text{ or } a \sim b'\text{-}) \\
A_4 : a' \prec b \text{ and } a \prec b' \\
A_5 : \text{not } (b' \prec a') \\
A_6 : \text{not } (b \prec a') \text{ and not } (b' \prec a)'
\end{array}
$$

Again, there are some obvious implications. But before we summarize these, let us show the following.

Lemma 4.6

$$A_4 \Leftrightarrow A_1.$$

Proof. $A_1 \Rightarrow A_4$ is obvious. Now let A_4 hold and $a \prec b$, $a' \sim a$, $b' \sim b$. In a first step, we conclude $a' \prec b$ and using A_4 a second time we have $a' \prec b'$ as desired for A_1. □

Now we may summarize the implications we have so far.

Diagram 4.7

$$
\begin{array}{ccccccc}
T_8 & \Leftarrow & T_4 & \Rightarrow & T_5 & \Rightarrow & T_6 \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
T_7 & \Leftarrow & T_1 & \Rightarrow & T_2 & \Rightarrow & T_3 \\
 & & \Downarrow & & & & \\
 & & A_6 & \Leftarrow & A_3 & \Leftarrow & A_4 \\
 & & \Uparrow & & \Uparrow & & \Updownarrow \\
 & & A_5 & \Leftarrow & A_2 & \Leftarrow & A_1
\end{array}
$$

Basically we will study pairs of T's and A's for defining "rational preference structures". So far we have 40 of them. In the next chapter we try to reduce this number. But before doing so, we want to present some interesting properties of A's and T's without further comments.

Lemma 4.8
If A_3 holds, then

$$a \prec b,\ a' \sim a,\ b' \sim b \Rightarrow a' \prec b' \text{ or } a' \sim b' \text{ or } (a' \sim b \text{ and } b' \sim a)$$

Proof. If A_3 holds and $a \prec b$, $a' \sim a$, $b' \sim b$ then ($a' \prec b'$ or $a' \sim b$) and ($a \prec b'$ or $a \sim b'$).

Applying A_3 again we obtain:

$$(a' \prec b' \text{ or } a' \sim b' \text{ or } a' \sim b) \text{ and } (a' \prec b' \text{ or } a' \sim b \text{ or } a \sim b')$$

Using standard propositional logic, we finally obtain

$$a' \prec b' \text{ or } a' \sim b' \text{ or } (a' \sim b \text{ and } a \sim b'). \ \square$$

This shows that $A_3 \Rightarrow A_2$ does not hold in general.

Lemma 4.9
If A_2 holds, then

$$a \lozenge b,\ a' \sim a,\ b' \sim b \Rightarrow a' \lozenge b' \text{ or } a' \sim b'.$$

Proof. Let A_2 hold and $a \lozenge b$, $a' \sim a$, $b' \sim b$. Now assume $a' \prec b'$. By A_2 we have (changing the role of a' and a, respectively b' and b) $a \prec b$ or $a \sim b$. Both cannot be true since $a \lozenge b$, Similarly we exclude $b' \prec a'$. The only possibilities left therefore are $a' \lozenge b'$, and $a' \sim b'$. □

Lemma 4.10
If A_1 holds, then

$$a \sim b,\ b \sim c \Rightarrow a \sim c \text{ or } a \lozenge c$$

Proof. Let A_1 hold and $a \sim b$, $b \sim c$. If now $a \prec c$, then $b \prec b$ and if $c \prec a$ then again $b \prec b$ which is not the case. Therefore $a \sim c$ or $a \lozenge c$. □

This shows how strong A_1 is: It makes $\sim$ "almost" transitive.

Lemma 4.11
If any A_i, T_4 or T_1 holds, then

$$a \sim b \Rightarrow \text{ not } (a \prec c \text{ and } c \prec b) \text{ and not } (b \prec c \text{ and } c \prec a) \text{ for all } c \in A.$$

Proof. By diagram 4.7. we know that it is sufficient to prove the implication above just for A_6. Assume A_6 holds and $a \sim b$, $a \prec c$ and $c \prec b$. Applying A_6 to $a \sim b$, $a \prec c$ we obtain $\text{not}(c \prec b)$ which contradicts the assumption $c \prec b$. The second part follows analogously. □

5 Rationality of Preference Structures

The basic idea of defining *rational preference structures* is to select on or more pairs of A's and T's. There may be "semantical" reasons for choosing some pair. In this paper we want to stress structural or "syntactical" reasons to this end.

So far we have 40 such pairs. In [ ] we have proved several results of the form

$$\text{Lemma: If } A_k \text{ holds, then}$$
$$T_i \Leftrightarrow T_j.$$

Doing this and insisting on transitivity of $\prec$ (i.e. T_1), we were able to reduce the list to 14 candidates, which is still too much. We then introduced an additional argument to reduce this number further. In this paper we reverse the order of arguments and obtain a good result much simpler. Let us include first just one result of the above type for illustration.

Lemma 5.1
If any A_i holds, then $T_1 \Leftrightarrow T_2$.

Proof. If any A_i holds, then A' : $a \prec b$, $a' \sim a \Rightarrow$ not $(b \prec a')$, since $A_i \Rightarrow A_6 \Rightarrow A'$. We know $T_1 \Rightarrow T_2$ and have to show $T_2 \Rightarrow T_1$ under the assumption of A'.
Assume $a \prec b, b \prec c$. By T_2 we conclude $a \prec c$ or $a \sim c$. If $a \sim c$, then $\text{not}(b \prec c)$ by A'. This is not true by assumption and hence $a \prec c$ as necessary for T_1. □

Some authors have used properties of the relation $\precsim$ (with the obvious meaning of "$\prec$ or $\sim$") to define useful properties of A = (A; $\prec, \sim$). Our next result shows, that at least there is no easy way using $\precsim$ for our purpose.

Lemma 5.2
$(A;\precsim)$ is a partially ordered set iff $(A; \prec, \sim)$ is crisp and A_1, T_1 hold.

Proof.

1. Assume, $(A;\precsim)$ is a partially ordered set and $a \sim b$, $b \sim c$. Clearly $a \precsim b$, $b \precsim c$ and therefore $a \precsim c$ by assumption. At the other hand also $b \precsim a$ and $c \precsim b$ and therefore $c \precsim a$. This means $a \sim c$ in partially ordered sets. Therefore, $(A; \prec, \sim)$ is *crisp*.

Now let $a \prec b$, $a' \sim a$, $b' \sim b$. From this we have $a \precsim b$, $b \precsim b'$, $a' \precsim a$ and by transitivity of $\precsim$ we obtain $a \precsim b'$, $a' \precsim a$. Again from transitivity finally $a' \precsim b'$ which means $a' \prec b'$ or $a' \sim b'$. Therefore A_2 holds. But if $a' \sim b'$ then $a \sim b'$ and finally $a \sim b$ by A being crisp. This contradicts $a \prec b$ and therefore $a' \prec b'$ and A_1 holds.

Finally assume $a \prec b, b \prec c$ and therefore a $\precsim$ b, b $\precsim$ c. Since $\precsim$ is transitive, we conclude a $\precsim$ c i.e. $a \prec c$ or $a \sim c$. Hence T_2 holds and by lemma 5.1 also T_1 holds, since we have already shown A_1.

2. Now let $(A; \prec, \sim)$ be crisp and A_1, T_1 hold. Assume a $\precsim$ b, b $\precsim$ c. We consider four cases.

(a) $a \sim b, b \sim c$. Since A is crisp, we have $a \sim c$ and hence $a \precsim c$.
(b) $a \prec c, b \sim c$. From A_1 we get $a \prec c$ and hence $a \precsim c$.
(c) $a \sim b$ $b \prec c$. Again from A_1 we have $a \prec c$ and hence $a \precsim c$.
(d) $a \prec b, b \prec c$. Here we conclude $a \prec c$ and hence $a \precsim c$ from T_1.

Finally we obtain $a \sim b$ from $a \precsim b$, $b \precsim a$ through lemma 3.5. □

If we would adopt A_1, T_1 for a definition of "rational", we would exclude Example 4.1. Therefore assuming $(A; \precsim)$ being a partially ordered set is too strong. The following observation will take us further.

Observation 5.3
If $\sim$ is the identity, then
1. All T's are equivalent to T_1.
2. All A's are obsolete.

Hence:

If $\sim$ is the identity, then $A = (A; \prec, \sim)$ is rational iff $(A; \precsim)$ is a partially ordered set. Idea: "eliminate" $\sim$ or "reduce" it to the identity.

The following technique of "reducing" is taken from the reduction of incomplete finite automata. See [ ] for many references. It also plays some role in clustering, as is explained in [ ].

Definition 5.4
Let $\sim$ be a similarity on A.

1. $B \subseteq A$ is a <u>subcluster</u> (with respect to $\sim$) iff $b_1 \sim b_2$ for any $b_1, b_2 \in B$.
2. $B \subseteq A$ is a <u>cluster</u> (with respect to $\sim$) iff
(a) B is a subcluster
(b) $B \subseteq B', B \neq B'$, then B' is not a subcluster.

Clusters are called "maximal compatibility classes" in [ ]. Our program now is, to study properties of the set of all clusters in A with respect to $\sim$.

Lemma 5.5
Let $\mathcal{A} = (A; \prec, \sim)$ be a preference structure. If P_1 and P_2 are clusters, then not ($b_1 \sim b_2$) for some $b_1 \in P_1, b_2 \in P_2$ or $P_1 = P_2$.

Proof. If $b_1 \sim b_2$ for some $b_1 \in P_1$ and all $b_2 \in P_2$, then $b_1 \in P_2$. Therefore either $P_1 = P_2$, or P_2 is not maximal. □

This means that if $P_1 \neq P_2$ then there are always $b_1 \in P_1, b_2 \in P_2$ such that $b_1 \prec b_2, b_2 \prec b_1$ or $b_1 \lozenge b_2$. This allows the following definition.

Definition 5.6
Let $(A; \prec, \sim)$ be a preference structure and P_1, P_2 clusters with respect to ~.

$$P_1 \leq P_2 \quad \text{iff} \quad b_1 \prec b_2 \text{ for some } b_1 \in P_1, b_2 \in P_2 \text{ or } P_1 = P_2$$

Note that " $P_1 \leq P_2$ and $P_2 \leq P_1$ and $P_1 \neq P_2$ " is possible. Also possible is "neither $P_1 \leq P_2$ nor $P_2 \leq P_1$", but then necessarily $b_1 \lozenge b_2$ for some $b_1 \in P_1$ and $b_2 \in P_2$ and $b'_1 \sim b'_2$ for all others. What we do have this obviously:

Lemma 5.7
If $P = \{P_1, \ldots, P_k\}$ is the set of all clusters of $\mathcal{A} = (A; \prec, \sim)$ then $\mathcal{P} = (P; <, id, \lozenge)$ is a weak preference structure where

$$P_i \lozenge P_j \Leftrightarrow \text{not } (P_i \leq P_j \text{ or } P_j \leq P_i)$$

Observation 5.3 tells us, that $\mathcal{P} = (P; <, id)$ is "rational" iff $(P; \leq)$ is a partially ordered set. The basic idea now is to call $\mathcal{A} = (A; \prec, \sim)$ "rational" iff $\mathcal{P} = (P; <, id)$ is rational, i.e. iff $(P; \leq)$ is a partially ordered set. Unfortunately, our $\mathcal{A}$'s and $\mathcal{T}$'s used so far cannot model this situation completely. But:

Lemma 5.8

$$(P_i \leq P_j, P_j \leq P_i \Rightarrow P_i = P_j) \text{ iff } A_5 \text{ holds.}$$

Proof.
1. Assume A_5 holds and $P_i \leq P_j$ as well as $P_j \leq P_i$ and $P_i \neq P_j$. There exist $a, a' \in P_i$ and $b, b' \in P_j$ such that $a \prec b, b' \prec a'$. But $a \sim a'$ and $b \sim b'$ gives us $\text{not}(b' \prec a')$ by A_5, a contradiction. Hence if $P_i \leq P_j$ and $P_j \leq P_i$, then $P_i = P_j$.

2. Assume, $(P_i \leq P_j, P_j \leq P_i \Rightarrow P_i = P_j)$ holds and $a \prec b$, $a' \sim a$, $b' \sim b$. If $a \in P_i$ and $b \in P_j$, we know $P_i \leq P_j$ and $P_i \neq P_j$. Therefore $\text{not}(P_j \leq P_i)$ and $\text{not}(b' \prec a')$ as wanted for A_5. □

Lemma 5.9
$(P_i \le P_j, P_j \le P_k \Rightarrow P_i \le P_k)$ iff T_{10} holds, where

$$T_{10} : a \prec b,\ c \prec d,\ b \sim c,\ a \in P_i,\ d \in P_j \Rightarrow a' \prec d' \text{ for some } a' \in P_i,\ d' \in P_j$$

Proof.
1. Assume T_{10} holds and $P_i \le P_j, P_j \le P_k$. This means $a \prec b$, $c \prec d$, $b \sim c$ for some $a \in P_i, c, d \in P_j, d \in P_k$. T_{10} guarantees $a'' \prec d''$ for some $a'' \in P_i, d'' \in P_j$. Hence $P_i \le P_k$ as desired.

2. Assume $\le$ is transitive and $a \prec b$, $c \prec d$, $b \sim c$, $a \in P_i$, $d \in P_k$. Since $b \sim c$, there is some P_j such that $b, c \in P_k$ and $P_i \le P_j, P_j \le P_k$. Hence $P_i \le P_k$ by transitivity. Hence $a'' \prec d''$ for some $a'' \in P_i, d'' \in P_k$ as wanted for T_{10}. □
Remark: a and d may be members of more than one P_i and P_j. There have to be a's and d's in <u>all</u> of them !
As a corollary we have the wanted result.

Theorem 5.10
$(P;\le)$ is a partially ordered set iff A_5 and T_{10} hold.

Therefore we would like to consider only pairs of A's and T's, which imply A_5 and T_{10}. To do so, we need the following obvious result.

Lemma 5.11

$$T_4 \Rightarrow T_{10} \Rightarrow T_8.$$

If we now check diagram 4.7, we conclude that the following pairs are left.

Diagram 5.12

$$\begin{array}{ccccc} A_1T_4 & \Rightarrow & A_2T_4 & \Rightarrow & A_5T_4 \\ \Downarrow & & \Downarrow & & \Downarrow \\ A_1T_{10} & \Rightarrow & A_2T_{10} & \Rightarrow & A_5T_{10} \end{array}$$

We can even reduce this number of pairs using the following results.

Lemma 5.13
1. If A_2 holds, then $T_{10} \Leftrightarrow T_4$.
2. If A_1 holds, then $T_{10} \Leftrightarrow T_1$.

Proof.
1. We know $T_4 \Rightarrow T_{10}$. Let therefore A_2 and T_{10} hold and $a \prec b$, $c \prec d$, $b \sim c$. By T_{10} we conclude $a' \prec d'$ fore some a', d' such that $a \sim a'$, $d \sim d'$. By A_2 we obtain $a \prec d$ or $a \sim d$.

Assume a $\sim$ d. From a $\prec$ b, b $\sim$ c, a $\sim$ d we conclude $d \prec c$ or d $\sim$ c by A_2 again. Both cannot be true since $c \prec d$. Hence $a \prec d$ and T_4 holds.

2. If A_1 holds, so does A_2. We have just shown $T_{10} \Leftrightarrow T_4$ for this case. Hence we only have to show $T_1 \Leftrightarrow T_4$ if A_1 holds. $T_4 \Rightarrow T_1$ is trivial and always true. Lets assume $a \prec b, c \prec d$ and $b \sim c$ and T_1. By A_1 we obtain $a \prec c$ and by T_1 we have $a \prec d$ as desired. □

Corollary 5.14
1. A_1T_4 holds iff A_1T_1 holds.
2. A_1T_{10} holds iff A_1T_1 holds.
3. A_2T_{10} holds iff A_2T_4 holds.

Therefore diagram 5.12 reduces to:

Diagram 5.15

$$A_1T_1 \Rightarrow A_2T_4 \Rightarrow A_5T_4 \Rightarrow A_5T_{10}$$

We know, that T_4 implies T_1, a desirable property of *rational*. A_5T_{10} does not. Hence we hesitate to call A_5T_{10} "rational". To get more feeling for A_5T_{10} (and for a later result) let's state the following.

Lemma 5.16
If A_5 holds, then $T_4 \Rightarrow T_{10} \Rightarrow T_{4'}$, where

$$T'_4 : a \prec b,\ c \prec d,\ b \sim c \Rightarrow a \prec d \text{ or } a \lozenge d.$$

Proof. $T_4 \Rightarrow T_{10}$ holds in general according to lemma 5.11. Let now T_{10} and A_5 hold. From a $\prec$ b, c $\prec$ d, b $\sim$ c, we conclude $a' \prec d'$ for some a' $\sim$ a, d' $\sim$ d by T_{10}. By A_5 we have $\text{not}(d \prec a)$, i.e. $a \prec d$ or a $\sim$ d or $a \lozenge d$. Assume. a $\sim$ d From a $\prec$ b, b $\sim$ c, a $\sim$ d we obtain $\text{not}(c \prec d)$ through A_5. But we know $c \prec d$ and therefore a $\sim$ d does not hold and we have proven T'_4. □

So far, that's all we can do. Let's proceed to our definitions.

6 Proposed Definitions

We have systematically reduced the number of candidates for *rational preference structures* to four, using structural arguments only. It seems reasonable to consider all four candidates. This leads to the following definitions.

Definition 6.1
The preference structure $(A; \prec, \sim)$ is

1. strongly rational : $\Leftrightarrow$
A_1 : $a \prec b, a' \sim a, b' \sim b \Rightarrow a' \prec b'$
T_1 : $a \prec b, b \prec c \Rightarrow a \prec c$

2. rational : $\Leftrightarrow$
A_2 : $a \prec b, a' \sim a, b' \sim b \Rightarrow a' \prec b'$ or $a' \sim b'$
T_4 : $a \prec b, c \prec d, b \sim c \Rightarrow a \prec d$

3. weakly rational : $\Leftrightarrow$
A_5 : $a \prec b, a' \sim a, b' \sim b \Rightarrow$ not $(b' \prec a')$
T_4 : $a \prec b, c \prec d, b \sim c \Rightarrow a \prec d$

4. proto-rational : $\Leftrightarrow$
A_5 : $a \prec b, a' \sim a, b' \sim b \Rightarrow$ not $(b' \prec a')$
T_{10} : $a \prec b, c \prec d, b \sim c, a \in P_i, d \in P_j \Rightarrow a' \prec d'$ for some $a' \in P_i$, $d' \in P_j$
where P_i, P_j are clusters with respect to $\sim$

Results of section 4 and 5 now become results about rationality if we start with definition 6.1.

Lemma 6.2
1. If A is strongly rational, then A is rational.
2. If A is rational, then A is weakly rational.
3. If A is weakly rational, then A is proto-rational.

Lemma 6.3
If A is weakly rational, then $\prec$ is transitive (but not necessarily $\precsim$).

Lemma 6.4
A is proto-rational iff $(P, \leq)$ is a partially ordered set.

Lemma 6.5
If A is rational, then $a \lozenge b, a' \sim a, b' \sim b \Rightarrow a' \lozenge b'$ or $a' \sim b'$.

Lemma 6.6
If A is strongly rational, then $a \sim b, b \sim c \Rightarrow a \sim c$ or $a \lozenge b$.

From the structure of T_1, A_1, T_4, A_2, A_5 we conclude easily:

Lemma 6.7
If A is (strongly, weakly) rational, then every subset of A is (strongly, weakly) rational.

This is not true for proto-rational preference structures.

Lemma 6.8
A is strongly rational iff A_4 and T_7 hold.

Proof. We know $A_1 \Leftrightarrow A_4$ from 4.6. We also know $T_1 \Rightarrow T_7$ from diagram 4.3. Hence we have to show $T_7 \Rightarrow T_1$ assuming A_1.

From $a \prec b, b \prec c$ we conclude $a' \prec b'$ for some a' ~ a, c' ~ c from T_7. Using A_1 we get $a \prec b$ as desired. □

In $[\ ]$ we have shown, that in fact <u>all</u> T_i's are equivalent, if we assume A_1.

A final remark in this section: We still have to prove that non of the deductions of 6.2 can be reversed, or in other words that no two of our definitions in fact do coincide. This can be done by giving examples that discriminate between them.

In order to make clearer where we have to search for those examples, we will first study some special cases.

7 Special Cases

In this chapter, we want to explore, to what extent our definitions meet intuition in special cases. The cases we want to consider are: $\sim$ is the identity, A is crisp (i.e. $\sim$ is transitive) and A is total.

As we have mentioned before, if $\sim$ is the identity, then all A's are equivalent to A_1 and all T's are equivalent to T_1. Hence the following is no surprise.

Lemma 7.1
If $A = (A; \prec, \mathrm{id})$, then the following statements are equivalent:

1. A is strongly rational
2. A is rational
3. A is weakly rational
4. A is proto-rational
5. $(A, \precsim)$ is a partial order

Our next special case is transitivity of $\sim$, i.e. A is crisp.

Lemma 7.2
If A is crisp, then $A_1 \Leftrightarrow A_2$

Proof. We know $A_1 \Rightarrow A_2$. If A_2 holds, and a $\prec$ b, a' ~ a, b'~ b then $a' \prec b'$ or a'~ b'. Assume a' ~ b' holds. Using transitivity of ~, we conclude a ~ b' and a ~ b. This cannot be true since $a \prec b$. Therefore $a' \prec b'$ and A_1 holds. □

Corollary 7.3
If A is crisp, the following statements are equivalent:

1. A is strongly rational
2. A is rational

Proof. From lemma 7.2 we have A_1T_1 for "strongly rational" and A_1T_4 "rational". But we know $T_4 \Rightarrow T_1$ which concludes the proof. □

Next we will show, that the definitions of "weakly rational" and "proto-rational" can be simplified considerably if A is crisp.

Lemma 7.4
If A is crisp, then $A_5 \Leftrightarrow A'_1$,
where A'_1: a $\prec$ b, a ~ a', b ~ b' $\Rightarrow$ a' $\prec$ b' or a' ◊ b'.

Proof. $A'_1 \Rightarrow A_5$ is obvious. If A_5 holds and a $\prec$ b, a' ~ a, b' ~ b then $\text{not}(b' \prec a')$, i.e. $a' \prec b'$ or a'~ b' or a' ◊ b'. Assume a' ~.b' As before, we can conclude a ~.b, which cannot be true. Hence A'_1 holds. □

Lemma 7.5
If A is crisp, then $T_8 \Leftrightarrow T_{10}$.

Proof. We know $T_{10} \Rightarrow T_8$. If A is crisp, then every $a \in A$ belongs to exactly one cluster with respect to ~. Therefore $T_8 \Rightarrow T_{10}$ by definition of T_{10}. □

Corollary 7.6
If A is crisp, then

1. A is weakly rational iff A'_1 and T_4 hold
2. A is proto-rational iff A'_1 and T_8 hold

Let us now come to the important special case where A is total.

Lemma 7.7
If A is total and proto-rational, then $(P, \leq)$ is a linear order.

Proof. (P, ≤) is a partially ordered set by definition. But $a \Diamond b$ does not happen and therefore $P_i \le P_j$ or $P_j \le P_i$ for all pairs of clusters.

Directly from lemma 6.6 we derive

Lemma 7.8
If A is total and strongly rational, then A is crisp.

Also trivially from the definitions we obtain:

Lemma 7.9
If A is total, then $A_5 \Leftrightarrow A_2$.

Corollary 7.10
If *A* is total, then the following statements are equivalent:

1. A is rational
2. A is weakly rational
3. A is proto-rational

Proof. We know $A_2T_4 \Rightarrow A_5T_4 \Rightarrow A_5T_{10}$ and have to show $A_5T_{10} \Rightarrow A_2T_4$. But $A_5T_{10} \Rightarrow A_2T_{10}$ by lemma 7.9 and $A_2T_{10} \Rightarrow A_2T_4$ by lemma 5.13. □

We have shown that in the case of total preference structures only two possibilities remain: strongly rational and rational!

Finally, the following result about a very special case follows directly from the above.

Lemma 7.11
If A is total <u>and</u> crisp, then the following statements are equivalent:

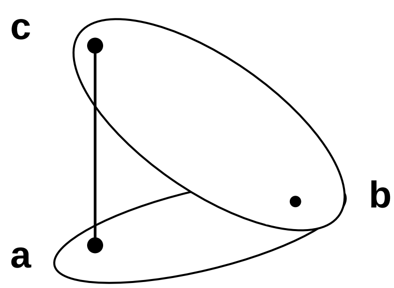
1. A is strongly rational
2. A is rational
3. A is weakly rational
4. A is proto-rational
5. $(A, \precsim)$ is a linear order

8 Examples

We will use this section to demonstrate, that none of the notions of definition 6.1 is equivalent to one of the others. To discriminate beween "strongly rational" and "rational" we use a short version of example 4.10 and use an obvious graphical notation.

Example 8.1

A = {a, b, c}, a ~ b, b ~ c, a ≺ c‘



We have to show, that A_2 and T_4 hold, but not A_1. T_4 is obvious since there is only one pair $x \prec y$. For A_2 we have three cases:

1. a ≺ c, a ~ b ⇒ b ≺ c or b ~ c.
   o.k. since b ~ c

2. a ≺ c ,c ~ b ⇒ a ≺ b or a ~ b
   o.k. since a ~ b

3. a ≺c ,a ~ b, b ~ c ⇒ b≺ b or b ~ b ok. since b ~ b.

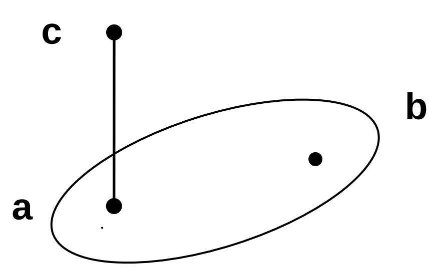
Any of the three cases contradicts A_1.

Note, that this example is total. From corollary 7.10 we know that there is no total preference structure to discriminate between the three others.

Next case is "rational" versus "weakly rational".

Example 8.2

A = {a, b, c}, a ~ b, a ≺ c

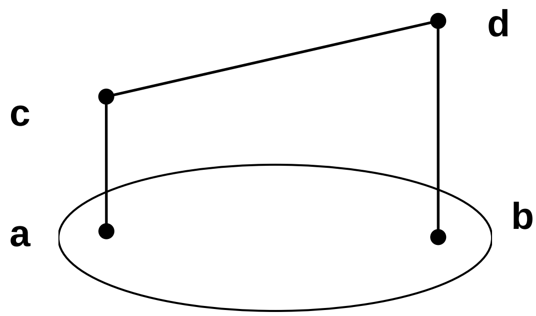


We have to show, that A_5 and T_4 hold, but A_2 does not. Again, T_4 is obvious, since there is only one pair $x \prec y$. A_5 has the form a ≺ c ,a ~ b ⇒ not (c ≺ b), which is fulfilled since b ◊ c. But A_2 has the form a ≺ c ,a ~ b ⇒ b ≺ c or b ~ c which is not fulfilled since b ◊ c.

Next case is "weakly rational" versus "proto-rational".

Example 8.3

A = {a, b, c, d}, $a \sim b$, $a \prec c$, $b \prec d$, $c \prec d$



We have to show that A_5 and T_{10} hold, but T_4 does not. For A_5 we have two situations.

1. $a \prec c, b \sim a \Rightarrow$ not $(c \prec b)$
   o.k. since $c \lozenge b$

2. $b \prec d, b \sim a \Rightarrow$ not $(d \prec a)$
   o.k. since $a \lozenge d$.

For T_{10} let $P_1 = \{a, b\}, P_2 = \{d\}$. The only situation for T_{10} is:

$$a \prec c, c \prec d, a \in P_1, d \in P_2 \Rightarrow b \prec d, b \in P_1, d \in P_2,$$

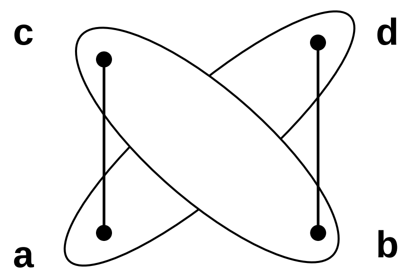
which is fulfilled. But from T_4 applied to the same situation, we would obtain $a \prec d$ which is not true since $a \lozenge d$.

If we use the identity for ~, we know that there are strongly rational as well as not even proto-rational preference relations, simply because there are partially ordered sets and sets that are not partially ordered.

Our last examples show, that there are preference relations that are not proto-rational with non-trivial similarities.

Example 8.4

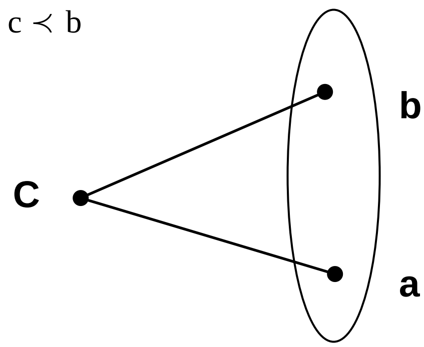
A = {a, b, c, d} $b \sim c$, $a \sim d$, $a \prec d$, $c \prec b$



From A_5 we would have $a \prec c$, $a \sim d$, $b \sim c \Rightarrow$ not $(c \prec d)$, which is obviously wrong.
Directly from lemma 4.11 we obtain the following counterexample.

Example 8.5

A = {a, b, c, } $a \sim b$, $a \prec c$, $c \prec b$



9 Utilities and Rationality

In chapter 2 we have briefly described the classical theory of preferences and then developed an axiomatic theory without any relation to this classical theory. We will now show, to what extend the two theories are related.

Definition 9.1

Let U = (U, ≤) be a partially ordered set, A a finite set and $\upsilon : A \to U$.
U is called an <u>utility</u> and υ an <u>utility function.</u>

We can use υ to construct some preference structure on A.

Definition 9.2

Let $\upsilon : A \to U$ be a utility function and $a, b \in A$.

$a \sim b :\Leftrightarrow \upsilon(a) = \upsilon(b)$

$a \prec b :\Leftrightarrow \upsilon(a) \leq \upsilon(b)$ and not $(a \sim b)$

$(A; \prec, \sim) = g(A, U, \leq, \upsilon)$

Theorem 9.3
If A = g(A, U, $\leq$, υ) then A is a crisp and strongly rational preference structure.

Proof.

By lemma 5.2 it is sufficient to show that (A; $\precsim$) is a partially ordered set.

1. a $\precsim$ a.

Obviously, $\upsilon(a) = \upsilon(a)$. But then by definition a ~ a and hence a $\precsim$ a.

2. a $\precsim$ b, b $\precsim$ a $\Rightarrow$ a ~ b.

From a $\precsim$ b, b $\precsim$ a we have $\upsilon(a) \leq \upsilon(b)$, $\upsilon(b) \leq \upsilon(a)$.Sinde (V; $\leq$) is partially ordered, we conclude $\upsilon(a) = \upsilon(b)$ and hence a ~ b.

3. a $\precsim$ b, b $\precsim$ c $\Rightarrow$ a $\precsim$ c

If a $\precsim$ b, b $\precsim$ c then $\upsilon(a) \leq \upsilon(b)$, $\upsilon(b) \leq \upsilon(c)$. Transitivity of $\leq$ results in $\upsilon(a) \leq \upsilon(c)$ and hence a $\precsim$c.

This completes the proof of theorem 9.3 □

Question: Are there any crisp, strongly rational preference structures, that <u>cannot</u> be obtained by some utility function ?

Our next result answers this question, since "strongly rational" implies "proto rational".

Theorem 9.4
If A is crisp and proto rational, then there is some utility U and some υ: A $\rightarrow$ U, such that A = g (A, V ,$\leq$, υ).

Proof. From chapter 5 we know, that U = (P, $\leq$) is a partially ordered set, where P are the equivalence classes of ~ and $\precsim$ is given by definition 5.6.

Lets choose U as the utiliy. Now consider υ: A $\rightarrow$ U where $\upsilon(a) = P_i$ iff a $\in$ P_i. Since A is crisp, $P_i \cap P_j = \phi$ if i $\neq$ j and therefore υ is a well defined mapping.

Clearly a ~ b iff $\upsilon(a) = \upsilon(b)$ and a $\prec$ b and not (a ~ b). This proofs our theorem.□

In the light of example 4.1 (which we consider interesting and important) our results so far show, that the current definitions of "utility" and "utility functions" are too narrow. We will extend them in the following way.

Definition 9.5
Let A be a finite set and U = (U, ≤, ~), where (U, ≤) is a partially ordered set and ~ a similarity on U. Any υ: A → U is called a general utililty function and (U, ≤, ~) general utility.

Definition 9.6
Let U = U, ≤, ~) be a general utility, A a finite set and υ: A → U.

g(A, U, ≤, ~, υ) = (A; ≺, ~) where:
a ~ b :⇔ υ(a) ~ υ(b)
a ≺ b: ⇔ υ(a) ≤ υ(b) and not (a ~ b)

Remark: υ(a) ~ υ(b) includes υ(a) = υ(b), since ~ is a similarity.

Theorem 9.7
g(A, U,≤, ~, υ) is a preference structure for which T_2 holds.

Proof.
1. Obviously υ(a) = υ(a), therefore υ(a) ~ υ(a) and hence a ~ a.

2. a ~ a implies υ(a) ~ υ(a). Clearly υ(b) ~ υ(a) and therefore b ~ a.

3. Assume a ≺ b, b ≺ a. for this we conclude υ(a) ≤ υ(b), υ(b) ≤ υ(a). Since (U, ≤) is partially ordered, we know υ(a) = υ(b), hence υ(a) ~ υ(b) and a ~ b. This is a contradiction. Hence a ≺ b implies not (b ≺ a).

4. If a ≺ b, then υ(a) ≤ (b) and not (a ~ b) by definition.

5. If a ≺ b, b ≺ c holds, we know by definition υ(a) ≤ υ(b), υ(b) ≤ υ(c).

This implies υ(a) ≾ υ(c), since U is partially ordered. But then a ≺ c or a ~ b as desired by T_2.□

Corollary 9.8
g(A, U, ≤, ~, υ) is strongly rational iff A_1 holds.

Proof. If g(A, U, ≤, ~, υ) is strongly rational, then A_1 holds by definition. If you know A_1 holds (or any A_i !) then T_2 = T_1. Hence g(A, U, ≤, ~, υ) is strongly rational by theorem 9.7. □

The question arises, how we can characterize "strongly rational", "rational" and "weakly rational" by properties of the given general utility. This is done in the following. First we introduce a special case.

Definition 9.9
$p(U, \leq \sim) := \quad g(U, U, \leq, \sim, id)$

Theorem 9.10
$g(A, U, \leq, \sim, \upsilon)$ is

1. strongly rational iff $p(U, \leq, \sim)$ is so
2. rational iff $p(U, \leq, \sim)$is so
3. weakly rational iff $p(U, \leq, \sim)$ is so.

Proof. We have to show that A_1, A_2, A_5, T_1 and T_4 hold for $g(A, U, \leq, \sim, \upsilon)$ iff it does so for $p(U, \leq, \sim)$, which straight forward by the definition of those preference structures. □

To proove an equivalent to theorem 9.4 in the non-crisp case ist not so easy, since we cannot use ~ itself. Let us consider the following refinement of ~.

Definition 9.11
Let $A = (A; \prec, \sim)$ be a preference structure.

$a \approx b :\Leftrightarrow$ 1. $a \sim b$ and
2. $a \sim c$ iff $b \sim c$ for all $c \in A$.

Lemma 9.12
$\approx$ is an equivalence relation on A.

Proof. $a \approx a$ and ($b \approx a \Rightarrow a \approx b$) is obvious.

Now assume $a \approx b$, $b \approx c$. By definition this means

1. $a \sim b$, $b \sim c$
2. ($a \sim d \Leftrightarrow b \sim d$) and ($b \sim e \Leftrightarrow c \sim e$) for all $d, e \in A$.
   If we take a for e, we conclude ($b \sim a \Leftrightarrow c \sim a$) and hence $c \sim a$.

Now consider some $f \in A$. We have ($a \sim f \Leftrightarrow b \sim f$) and ($b \sim f \Leftrightarrow c \sim f$). From this we obtain ($a \sim f \Leftrightarrow c \sim f$) and hence $\approx$ is transitive. □

As mentioned before, $\approx$ is a refinement of ~. When are these relations equal ?

Lemma 9.13

$\approx$ is equal to $\sim$ iff A is crisp.

Proof. 1. Let $a \sim b$ and $b \sim c$. If $\approx$ is equal to $\sim$, this means $a \approx b$, $b \approx c$. Since $\approx$ is transitive, we have $a \approx c$ and $a \sim c$. Therefore $\sim$ is transitive and A crisp.

2. Now let A be crisp and $a \sim b$. Assume further that $a \sim c$ for some $c \in A$. Since A is crisp we conclude $b \sim c$ and vice versa. Therefore $a \approx b$, as we wanted to show. □

The relation $\approx$ is now used to prove the following result about the generation of preference structures. It is a generalization of theorem 9.4.

Theorem 9.14

If A is weakly rational, then there is some general utility $(U, \leq, \sim)$ and $\upsilon: A \rightarrow U$, sucht that $A = g(A, U, \leq, \sim, \upsilon)$.

Proof. Let $\{U_1, .., U_k\} = U$ be the equivalence classes of $\approx$ on A. Now we define

1. $U_i \leq U_j :\Leftrightarrow u_i < u_j$ for some $u_i \in U_i$, $u_j \in U_j$ or $i = j$

2. $U_i \sim U_j :\Leftrightarrow u_i \approx u_j$ for some $u_i \in U_i$, $u_j \in U_j$.

We have to show first, that $(U, \leq, \sim)$ is a utility.

1. $U_i \leq U_i$ is obvious. Let $U_i \leq U_j$ hold. Hence $u_i \prec u_j$ for some $u_i \in U_i$ and $u_j \in U_j$. Take $u'_i \in U_i$ and $u'_j \in U_j$. Clearly $u_i \approx u'_i$ and $u_j \approx u'_j$ and hence $u_i \sim u'_i$ and $u_j \sim u'_j$. Since A is weakly rational, A_2 holds. Therefore $u'_i \prec u'_j$ or $u'_i \sim u'_j$ hold and $U_j \leq U_i$ can only be true if $i = j$. Now $U_i \leq U_j$, $U_j \leq U_k$. We know $u_i \prec u_j$, $u_j \prec u_k$ for some $u_i \in U_i$, u'_j, $u_j \in U_j$ and $u_k \in U_k$. From u'_j, $u_j \in U_j$ we know $u'_j \sim u_j$ and from T_8 we conclude that $\bar{u}_i \prec \bar{u}_k$ for some $\bar{u}_i \in U_i$, $\bar{u}_k \in U_k$. But this means U_i, $\leq U_k$ and $\leq$ is a partial order on U.

2. Let $U_i \sim U_j$ hold, i. e. $u_i \approx u_j$ for some $u_i \in U_i$, $u_j \in U_j$. Consider $u'_i \in U_i$, $u'_j \in U_j$. This means $u_i \approx u'_i$, $u_j \approx u'_j$. We know, that $\approx$ is transitiv, therefore: $u_i \approx u_j$ and $u'_i \approx u_i$, implies $u'_i \approx u_j$. This together with $u'_j \approx u_j$ gives us $u'_i \approx u'_j$. Hence $\sim$ is well defined.

$U_i \sim U_j$ and $(U_j \sim U_j \Rightarrow U_j \sim U_i)$ is obvious. Hence $\sim$ is a similarity on U.

Next we have to define some u: $A \rightarrow U$. We use the same idea as in the proof of theorem 9.4 .

$\upsilon(a) = U_i :\Leftrightarrow a \in U_i$. Clearly $A = g(A, U, \leq, \sim, u)$. □

Another property of $g(A, U, \leq, \sim, u)$ may be of interest especially in the light of examplee 4.1. Let's start with the following general result.

Lemma 9.15
If A = g(A, U, $\leq$, $\sim$, u) and T_1 or any A_i hold, then:

$a \sim c,\ a \precsim b \precsim c \Rightarrow a \sim b$ <u>or</u> $b \sim c$

Proof. We know from theorem 9.7 that T_2 holds. Lemma 5.1 gives us T_1. Now assume $a \sim c$, $a \precsim b \precsim c$ and not $(a \sim b)$, not $(b \sim c)$. Then clearly $a \prec b$, $b \prec c$ which by T_1 implies $a \prec c$. This is a contradiction to $a \sim c$ and hence $a \sim b$ or $b \sim c$. □

Corollary 9.16
If A = g(A, U, $\leq$, $\sim$, u) is crisp and T_1 or any A_i hold, then

$a \sim c,\ a \precsim b \precsim c \Rightarrow a \sim b$ <u>and</u> $b \sim c$.

Proof. We know, that $a \sim b$ <u>or</u> $b \sim c$ holds. If $\sim$ is transitive, then $a \sim c$, $a \sim b$ implies $a \sim b$ and $a \sim c$. $b \sim c$ implies $b \sim c$. □

This property will be used as special case.

Definition 9.17
(A, I, $\leq$, $\sim$, u)A = g(A, U, $\leq$, $\sim$, u) is called <u>interval-based</u> iff

$a \precsim b \precsim c,\ a \sim c \Rightarrow a \sim b$ <u>and</u> $b \sim c$ for all $a, b, c \in U$.

Example 4.1 is interval-based.

The final subject we want to study in this chapter is the special case of U = I where I is some linear order. In the following I always denotes such a linear order.

We observe that g(A, I, $\leq$, $\sim$, u) is always total. Hence the related results of chapter 7 apply. Therefore:

Lemma 9.18
A_1 holds for g(A, U, $\leq$, $\sim$, u).

For interval-based preference structures ((A, I, $\leq$, $\sim$, u) we have the following strong result.

Theorem 9.19
If A = g(A, I, $\leq$, $\sim$, u) is interval-based, then A is rational.

Proof. It is sufficient to show, that p(I, $\leq$, $\sim$) is rational as we know from lemma 9.4.

1.To prove A_2 let $a \prec b$, $a' \sim a$, $b' \sim b$. If $a' \precsim b'$ then $a' \prec b'$ or $a' \sim b'$ as wanted. Let now $b' \precsim a'$. If $b' \precsim a \precsim b$ then $a \sim b$ (interval-based !) which is not true. If $a \precsim b \precsim a'$, then $b \sim a$ (interval-based !) which is also not true. Hence $a \precsim b' \precsim a' \precsim b$. Since $a' \sim a$, we conclude $a' \sim b'$ as desired. Therefore A_2 holds.

2. Now let $a \prec b$, $c \prec d$, $b \sim c$ be true. We conclude $a \precsim b$, $c \precsim d$. If $c \precsim a \precsim b$, then $a \sim b$ (interval-based !) which is not true. Hence $a \precsim c$.

If $b \precsim c$, then $a \precsim d$. Let $a \sim b$ be true. Then $c \sim d$ since $a \precsim b \precsim c \precsim d$, which is not true. Hence $a \prec d$ in this case, as desired. The only remaining case is $a \precsim c \precsim d \precsim b$, But then $d \sim c$ (interval-based !), which is not true. Hence T_4 holds and finally A is rational. □

We want to finish this chapter with a conjecture. If it turns out to be true, it shows the importance of example 4.1

Conjecture 9.20
If A is total and rational, then $A = g(A, I, \leq, \sim, \upsilon)$ for some I and υ and A is interval-based.

10 Conclusion and Outlook

We have given good structural reasons for axiomatic definitions of “strongly rational”, “rational”, “weakly rational” and “proto-rational”. The latter looks a bit complicated, but is justified by theorem 5.10. We have also shown, how these definitions relate to the classical theory of utility functions.

A rather rich body of results about these definitions were developed, especially for special cases that look very natural.

What else should be done ? Conjecture 9.20 should be proven or disproven ! The notions of homomorphisms, reductions, completions and compositions should be clarified. Last but not least, more general settings as indicated in section 3 should be explored.

Of special interest may be “fuzzy” extensions, as we want to indicate for “similarities”.

In our definition, either $a \sim b$ or not $(a \sim b)$ plus the two properties of “similarity”. In fuzzy theory, $a \sim b$ are similar to some degree, i. e. sim: $A \times A \rightarrow [0,1]$ and $a \sim b$ to the degree of sim (a, b).

Definition 10.1
sim is a fuzzy similarity on A iff

1.sim: $A \times A \rightarrow [0,1]$
2.sim (a, a) = 1
3.sim (a, b) = sim (b, a).

If we only look on similarity degrees of 1, we get the usual notion of "similarity".

Further more:

Definition 10.2
Let sim be a fuzzy similarity on A.

$\text{sim}_{\alpha}(a, b) = 1 :\Leftrightarrow \text{sim}(a, b) \geq \alpha$

$\text{sim}_{\alpha}(a\ b) = 0 :\Leftrightarrow \text{sim}(a, b) < \alpha$

Lemma 10.3
sim_{α} is a similarity on A for all $\alpha \in [0,1]$.

This may indicate, how to generalize other notions to fuzzy notions.