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***Abstract*—This paper defines Structured Markovian Arrival Processes (SMAPs). An SMAP consists of several blocks each being represented by a random variable specifying the duration of staying in that block. Leaving a block indicates an arrival event of the SMAP. The routing between blocks is governed by a stochastic matrix $\mathbf{Q}$. It is shown that the joint moments of the SMAP can be directly determined from the moments of the block random variables and routing matrix $\mathbf{Q}$, if $\mathbf{Q}$ is doubly stochastic. The characteristics of the SMAP can be computed very efficiently if $\mathbf{Q}$ is in addition circulant. Furthermore we show that for given block random variables the determination of a routing matrix $\mathbf{Q}$ and thus the fitting of the SMAP essentially results in solving a set of linear equations.**

I. Introduction

Since the seminal work of Neuts [19] Markovian Arrival Processes (MAPs) are commonly used for the description of complex arrival processes in Markov models. Especially their use in analytical models like queueing network models has been investigated intensely [8], [11], [12], [15], [23]. It is a considerable problem to find the parameters of a MAP so that the characteristics of an arrival process are captured accurately. Normally those characteristics are given as a sample (measured or from a simulation trace) or as specific statistical figures, like e.g. the joint moments of the interarrival times. Corresponding to these two forms of descriptions of an arrival process there are two major classes of fitting methods [16]: Fitting based on the sample data and fitting based on information extracted from the sample. An example of the first class of fitting methods is the expectation-maximisation method (e.g. [5], [22]). A drawback of this fitting method is that its computational complexity depends on the size of the sample data. In contrast, the second class of fitting methods offers the possibility to examine large datasets, since the calculation of derived characteristics (e.g. mean, variance etc.) is less computationally intensive. In this context several papers deal with the fitting of small MAPs usually consisting of a few states (e.g. [9], [10]), so that the state space of the overall performance model is still of manageable size, but with the drawback of covering only restricted forms of autocorrelations. Recently [21] presented a minimal representation of MAPs based on the joint moments of the interarrival time process and showed that properly chosen n^2 parameters are sufficient to determine a MAP with n states. Even more it is sufficient to consider moments of the interarrival time and joint moments of consecutive(!) interarrival times to capture also the long range behaviour and to characterise a MAP.

In spite of such results there are still open problems concerning the characterisation of MAPs. One problem concerns a canonical representation of MAPs and [2] suggests to look for specific structures of MAPs for a better understanding. In this paper we investigate such a specific structure. In contrast to the recommendation of [2] we do not concentrate on the level of the state space, but try to consider MAPs from a more abstract point of view: A MAP consists of several blocks represented by random variables specifying the duration of internal behaviour. Leaving a block indicates an arrival event and the routing between blocks is described by a routing matrix $\mathbf{Q}$. The main result of this paper expresses that the joint moments of the MAP can be determined from the moments of the block random variables and routing matrix $\mathbf{Q}$, if $\mathbf{Q}$ is doubly stochastic. A special class of doubly stochastic routing matrices are circulant matrices and we additionally show that for given block distributions the fitting of MAPs essentially results in solving a set of linear equations for determining a circulant routing matrix.

The outline of the paper is as follows. In Sect. II we present some basic definitions before we give the main result in Sect. III. Sect. IV considers circulant routing matrices.

II. Basic definitions

MAPs are usually defined by two square matrices $\mathbf{D_0}$ and $\mathbf{D_1}$ of the same order, such that the sum $\mathbf{D} = \mathbf{D_0} + \mathbf{D_1}$ is the generator of an irreducible Markov chain. Elements of matrix $\mathbf{D_1}$ describe transitions between states being associated with an arrival

and off-diagonal elements of matrix $\mathbf{D_0}$ are associated with internal transitions of the MAP. The steady state probability vector π is given by $\pi\mathbf{D} = \mathbf{0}, \pi\mathbf{1}^T = 1$ where $\mathbf{1} = (1, \ldots, 1)$ is a vector of ones. In this paper we will denote vectors $\mathbf{v}$ as row vectors and $\mathbf{v}^T$ and $\mathbf{C}^T$ denote the transpose of vector $\mathbf{v}$ and matrix $\mathbf{C}$, resp. We will use superscripts for matrices and vectors as follows: for a matrix $\mathbf{C}$, $\mathbf{C}^i$ is the i-th power of $\mathbf{C}$ and for a parameterised vector $\mathbf{v}$, $\mathbf{v}^i$ denotes the vector for parameter i and $\mathbf{v}^i(j)$ is the j-th component of $\mathbf{v}^i$.
The discrete time process embedded at arrival instants is given by the state transition probability matrix $\mathbf{P} = (-\mathbf{D_0})^{-1}\mathbf{D_1}$ and the corresponding steady state probability vector α is given by $\alpha\mathbf{P} = \alpha, \alpha\mathbf{1}^T = 1$.
In steady state, the distribution of the interarrival time X is given by $P[X < t] = 1 - \alpha e^{\mathbf{D_0}t}\mathbf{1}^T$. The moments of X and the joint moments of the interarrival time process are given by [15], [21]:

- k-th moment:

$$E[X^k] = k!\alpha(-\mathbf{D_0})^{-k}\mathbf{1}^T \tag{1}$$

- joint moments of the $0 = a_0 < a_1 < \ldots < a_k$-th interarrival times: $E[X_0^{i_0} X_{a_1}^{i_1} \ldots X_{a_k}^{i_k}] =$

$$\prod_{j=0}^{k} [i_j!] \;\; \alpha(-\mathbf{D_0})^{-i_0} \prod_{j=1}^{k} \left[\mathbf{P}^{(a_j - a_{j-1})}(-\mathbf{D_0})^{-i_j}\right] \mathbf{1}^T \tag{2}$$

[21] shows that the joint moments uniquely determine a MAP and that the corresponding matrix representations $(\mathbf{D_0}, \mathbf{D_1})$ are all similar.

III. Structured MAP

The main idea of this paper is to impose the following structure on the MAP. A structured MAP is defined here by a finite number of blocks $i, i = 1, \ldots, N$ and a routing matrix $\mathbf{Q} = (q(i,j)) \in \mathbb{R}_0^{+N \times N}$. Each block i is described by a random variable Y_i specifying the duration of staying in that block. Leaving block i indicates an arrival event and the process enters block j with probability $q(i,j)$. The situation is similar to a closed queueing network where a single customer moves around and leaving a station issues an arrival event. In this paper we assume that the Y_i have a hyper-Erlang distribution in order to simplify some notations. The choice of hyper-Erlang distributions is no restriction, since hyper-Erlang distributions can approximate general non-negative distributions arbitrarily closely [6], [18], [22].

Fig. 1 depicts the general structure of the MAP. The parameters of the hyper-Erlang distribution of block $i, i \in \{1, \ldots, N\}$ are

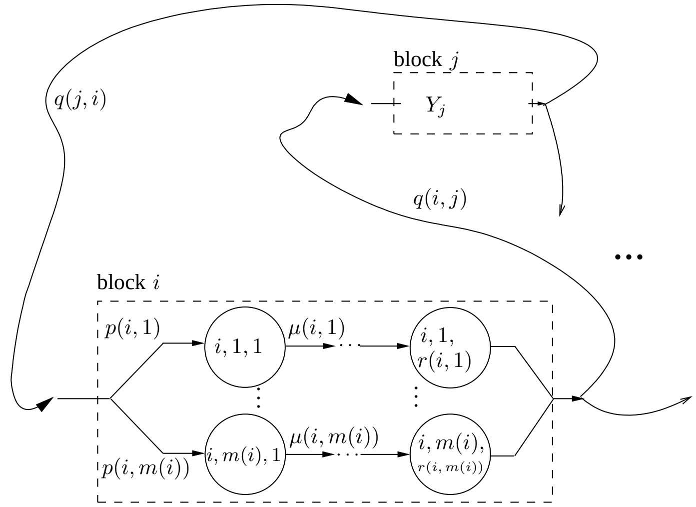

Fig. 1. Structured MAP

- $m(i) \in \mathbb{N}$ denoting the number of Erlang-branches.
- $r(i,b) \in \mathbb{N}$ is the number of states of branch b.
- $p(i,j) \in [0,1], j \in \{1, \ldots, m(i)\}$ denoting the probability of selecting branch j. Note that $\sum_{j=1}^{m(i)} p(i,j) = 1$.
- $\mu(i,j) \in \mathbb{R}^+$ is the parameter of the exponential distributions of branch j.

The moments of the hyper-Erlang distribution of block i are given by [22]

$$E[Y_i^k] = \sum_{b=1}^{m(i)} p(i,b) \frac{k!}{\mu(i,b)^k} \binom{k + r(i,b) - 1}{r(i,b) - 1} \tag{3}$$

and since the number of states is finite all moments do exist. For a concise notation we define $\mathcal{N} := \{1, \ldots, N\}$, $\mathcal{M}(i) := \{1, \ldots, m(i)\}$, $\mathcal{R}(i,b) := \{1, \ldots, r(i,b)\}$.

The state of the MAP is given by $(i,b,s), i \in \mathcal{N}$, $b \in \mathcal{M}(i), s \in \mathcal{R}(i,b)$ where i denotes the block number, b is the current branch of the hyper-Erlang distribution and s the corresponding current phase.

Definition 1 (SMAP): A **structured MAP (SMAP)** is a MAP given by $(N, \mathcal{B}, \mathcal{Y}, \mathbf{Q})$ where $N \in \mathbb{N}$ denotes the number of blocks, $\mathcal{B}$ with $| \mathcal{B} |= N$ is a set of blocks and $\mathcal{Y} = \{Y_1, \ldots, Y_N\}$ is a set of random variables with Y_i being the random variable for block i. $\mathbf{Q} = (q(i,j)) \in \mathbb{R}_0^{+N \times N}$ is an irreducible stochastic matrix whose entries $q(i,j)$ specify the probability of entering block j after having left block i.

In order to simplify notation, we use two auxiliary functions. Define for arbitrary ordered sets S function $\gamma : S \times S \mapsto \{0,1\}$ by

$$\gamma(i,j) := \begin{cases} 1 & \text{if } i < j \\ 0 & \text{otherwise} \end{cases}$$

and define for arbitrary sets S the discrete delta function $\delta : S \times S \mapsto \{0,1\}$ as

$$\delta(i,j) := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

With these definitions the matrices $\mathbf{D_0}, \mathbf{D_1}$ of an SMAP can be written as

$$\begin{aligned} \mathbf{D_0}((i,b,s),(j,c,t)) &= \delta((i,b),(j,c))\mu(i,b) \\ &\quad [\delta(s,t-1)\gamma(s,r(i,b)) - \delta(s,t)] \\ \mathbf{D_1}((i,b,s),(j,c,t)) &= \delta(t,1)\delta(s,r(i,b)) \\ &\quad q(i,j)p(j,c)\mu(i,b) \end{aligned} \tag{4}$$

As expected $\mathbf{D} := \mathbf{D_0} + \mathbf{D_1}$ is the generator of a Markov chain, since

$$\begin{aligned} &\sum_{j=1}^{N}\sum_{c=1}^{m(j)}\sum_{t=1}^{r(j,c)} \mathbf{D_0}((i,b,s),(j,c,t)) + \mathbf{D_1}((i,b,s),(j,c,t)) \\ &= \sum_{j=1}^{N}\sum_{c=1}^{m(j)}\sum_{t=1}^{r(j,c)} \Big\{ \delta((i,b),(j,c))\mu(i,b) \\ &\qquad [\delta(s,t-1)\gamma(s,r(i,b)) - \delta(s,t)] \\ &\qquad + \delta(t,1)\delta(s,r(i,b))q(i,j)p(j,c)\mu(i,b) \Big\} \\ &= \sum_{t=1}^{r(i,b)} \Big\{ [\delta(s,t-1)\gamma(s,r(i,b)) - \delta(s,t)]\,\mu(i,b) \Big\} \\ &\qquad + \delta(s,r(i,b)\mu(i,b) \sum_{j=1}^{N}\sum_{c=1}^{m(j)} q(i,j)p(j,c) \\ &= \sum_{t=1}^{r(i,b)} \Big\{ [\delta(s,t-1)\gamma(s,r(i,b)) - \delta(s,t)]\,\mu(i,b) \Big\} \\ &\qquad + \delta(s,r(i,b)\mu(i,b) = 0 \end{aligned}$$

The global balance equations $\pi(\mathbf{D_0} + \mathbf{D_1}) = \mathbf{0}$ are given by the following set of equations
$\forall i \in \mathcal{N}, b \in \mathcal{M}(i), s \in \mathcal{R}(i,b)$:

$$\pi(i,b,s) = \pi(i,b,s-1) \quad \text{if } s > 1 \tag{5}$$

$$\pi(i,b,s)\mu(i,b) = \sum_{k=1}^{N}\sum_{j=1}^{m(k)} \pi(k,j,r(k,j))\mu(k,j) \, q(k,i)p(i,b) \quad \text{if } s = 1 \tag{6}$$

In the following we will assume that the routing matrix $\mathbf{Q}$ is doubly stochastic [13]. A doubly stochastic matrix is also called bistochastic.

Definition 2: A matrix $\mathbf{Q} \in \mathbb{R}_0^{+N \times N}$ is doubly stochastic iff

$$\sum_{j=1}^{N} q(i,j) = \sum_{j=1}^{N} q(j,i) = 1, \quad \forall i \in \mathcal{N} \tag{7}$$

As we will see assuming $\mathbf{Q}$ to be doubly stochastic simplifies the calculation of the joint moments of the SMAP. Furthermore we can determine the steady state distribution of the Markov chain directly.

Theorem 1: Let $(N, \mathcal{B}, \mathcal{Y}, \mathbf{Q})$ be an SMAP with doubly stochastic matrix $\mathbf{Q}$, then the vector π satisfying $\pi\mathbf{D} = \mathbf{0}, \pi\mathbf{1}^T = 1$ is given by

$$\pi(i,b,s) = G\frac{p(i,b)}{\mu(i,b)} \tag{8}$$

$\forall i \in \mathcal{N}, b \in \mathcal{M}(i), s \in \mathcal{R}(i,b),$ with

$$G = [\sum_{i \in \mathcal{N}}\sum_{b=1}^{m(i)} r(i,b)\frac{p(i,b)}{\mu(i,b)}]^{-1} \tag{9}$$

G is a normalisation constant ensuring $\sum \pi(i,b,s) = 1$. Note that $\frac{1}{G} = \sum_i E[Y_i]$ where $E[Y_i]$ is the first moment (mean) of the hyper-Erlang distribution of block i.

Proof: of Th. 1:
Eq. (5) obviously holds, since (8) implies
$\pi(i,b,s) = \pi(i,b,t), \forall s,t \in \mathcal{R}(i,b)$.
Applying (8) to (6) gives

$$\begin{aligned} G\frac{p(i,b)}{\mu(i,b)}\mu(i,b) &= G\sum_{k=1}^{N}\sum_{j=1}^{m(k)} \frac{p(k,j)}{\mu(k,j)}\mu(k,j)q(k,i)p(i,b) \\ &= Gp(i,b)\sum_{k=1}^{N} q(k,i)\sum_{j=1}^{m(k)} p(k,j) \\ &= Gp(i,b) \end{aligned}$$

■

If $\mathbf{Q}$ is doubly stochastic, we will call an SMAP a **doubly stochastic SMAP**. The next theorem shows that the joint moments of doubly stochastic SMAPs can be directly calculated from the moments of the block distributions.

Define the vector $\mathbf{E}^k := \left(E[Y_1^k], E[Y_2^k], \ldots, E[Y_N^k]\right)$ where $E[Y_i^k]$ is the k-th moment of random variable Y_i. For a vector $\mathbf{v} \in \mathbb{R}^N$ let $Diag(\mathbf{v}) = (d(i,j)) \in \mathbb{R}^{N \times N}$ be the diagonal matrix with $d(i,j) = \delta(i,j)\mathbf{v}(i)$.

Theorem 2: Let $(N, \mathcal{B}, \mathcal{Y}, \mathbf{Q})$ be an SMAP with doubly stochastic matrix $\mathbf{Q}$, then the following holds for the joint moments of the $0 = a_0 < a_1 < \ldots < a_k$-th interarrival times:

$$E[X_0^{i_0} \ldots X_{a_k}^{i_k}] = \frac{1}{N}\mathbf{E}^{i_0}\prod_{j=1}^{k}\left[\mathbf{Q}^{m_j}Diag(\mathbf{E}^{i_j})\right]\mathbf{1}^T \tag{10}$$

with $m_j := a_j - a_{j-1}, j = 2, \ldots, k$, $m_1 := a_1$ and $\mathbf{1}$ being here an N-dimensional vector of ones.

Proof: We give an outline of the proof here. For details please see the appendix.

The expressions on the right hand side of (2) can be calculated directly using

$$\begin{aligned}(-\mathbf{D_0})^{-k}((i,b,s),(j,c,t)) &= \delta((i,b),(j,c))\gamma(s,t+1) \\ &\quad \mu(i,b)^{-k}\binom{k-1+t-s}{t-s} \\ \mathbf{P}^k((i,b,s),(j,c,t)) &= \delta(t,1)p(j,c)\mathbf{Q}^k(i,j) \\ \alpha(i,b,s) &= \frac{1}{N}\delta(s,1)p(i,b)\end{aligned}$$

and showing that for

$$\begin{aligned}\mathbf{v}_k &:= \prod_{j=1}^{k}[i_j!]\left[\mathbf{P}^{(a_j-a_{j-1})}(-\mathbf{D_0})^{-i_j}\right]\mathbf{1}^T \\ \mathbf{w}_k &:= \prod_{j=1}^{k}\left[\mathbf{Q}^{(a_j-a_{j-1})}Diag(\mathbf{E}^{i_j})\right]\mathbf{1}^T\end{aligned}$$

we have $\forall i \in \mathcal{N}$:

$$\mathbf{v}_k(i,b,s) = \mathbf{w}_k(i),\ \forall b \in \mathcal{M}(i), s \in \mathcal{R}(i,b)$$

which finally gives

$$[i_0!]\ \alpha(-\mathbf{D_0})^{-i_0}\mathbf{v}_k = \frac{1}{N}\mathbf{E}^{i_0}\mathbf{w}_k$$

■

It is worth noting that the expressions in (10) depend on the number of blocks and not on the number of states of the SMAP.

Corollary 1: For an SMAP $(N, \mathcal{B}, \mathcal{Y}, \mathbf{Q})$ with doubly stochastic matrix $\mathbf{Q}$, the following holds:
$\forall i, j, m \in \mathbb{N}_0$:

$$E[X^i] = \frac{1}{N}\mathbf{E}^i\mathbf{1}^T \tag{11}$$

$$E[X_0^iX_m^j] = \frac{1}{N}\mathbf{E}^i\mathbf{Q}^m\mathbf{E}^{jT} \tag{12}$$

Proof: Eq. (11) can be directly verified by calculating $E[X^i] = i!\alpha(-\mathbf{D_0})^{-i}\mathbf{1}^T$, it also follows from (10), using the convention $\prod_{j=1}^{0}\dots = \mathbf{I}$. Eq. (12) follows from (10) for $k = 1$, since $Diag(\mathbf{E}^j)\mathbf{1}^T = \mathbf{E}^{jT}$. ■

Note that Eq. (11) expresses that the i-th moment of X is given by the arithmetic mean of the i-th moments of the random variables of the blocks. Eq. (11) also shows that the moments of the interarrival time are independent of the routing matrix $\mathbf{Q}$ whereas Eq. (12) shows that the joint moments are determined by matrix $\mathbf{Q}$ for given block moments.

Example 1: Let $(N, \mathcal{B}, \mathcal{Y}, \mathbf{Q})$ be an SMAP with $N = 2, \mid \mathcal{B} \mid= 2, \mathcal{Y} = \{Y_1, Y_2\}, \mathbf{Q} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, where Y_1 is Erlang-2 distributed with parameters $m(1) = 1, r(1,1) = 2, p(1,1) = 1, \mu(1,1) = \lambda$ and Y_2 is hyper-exponentially distributed with parameters $m(2) = 2, r(2,1) = r(2,2) = 1, p(2,1) = 1 - p(2,2) = p, \mu(2,i) = \mu_i, i \in \{1,2\}$. Since

$$\begin{aligned}E[Y_1^k] &= \frac{(k+1)!}{\lambda^k} \\ E[Y_2^k] &= p\frac{k!}{\mu_1^k} + (1-p)\frac{k!}{\mu_2^k}\end{aligned}$$

the following holds for the SMAP:

$$\begin{aligned}E[X^k] &= \frac{1}{2}\left(\frac{(k+1)!}{\lambda^k} + p\frac{k!}{\mu_1^k} + (1-p)\frac{k!}{\mu_2^k}\right) \\ E[X_0^iX_1^j] &= \frac{1}{2}\left(\mathbf{E}^i(2)\mathbf{E}^j(1) + \mathbf{E}^i(1)\mathbf{E}^j(2)\right) \\ &= \frac{1}{2}\left(\left(p\frac{i!}{\mu_1^i} + (1-p)\frac{i!}{\mu_2^i}\right)\frac{(j+1)!}{\lambda^j}\right. \\ &\quad \left. + \left(p\frac{j!}{\mu_1^j} + (1-p)\frac{j!}{\mu_2^j}\right)\frac{(i+1)!}{\lambda^i}\right)\end{aligned}$$

If $\mathbf{Q}$ is NOT doubly stochastic, Th. 2 and Cor. 1 need not hold. E.g. if we change matrix $\mathbf{Q}$ from Ex. 1 to

$$\mathbf{Q} := \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix}$$

then Eq. (11) still gives

$$E[X] = \frac{1}{\lambda} + \frac{p}{2\mu_1} + \frac{1-p}{2\mu_2}$$

but evaluating (1) using

$$(-\mathbf{D_0})^{-1} = \begin{pmatrix} \frac{1}{\lambda} & \frac{1}{\lambda} & 0 & 0 \\ 0 & \frac{1}{\lambda} & 0 & 0 \\ 0 & 0 & \frac{1}{\mu_1} & 0 \\ 0 & 0 & 0 & \frac{1}{\mu_2} \end{pmatrix}$$

$$\text{and} \quad \alpha = \left(\frac{2}{3}, 0, \frac{1}{3}p, \frac{1}{3}(1-p)\right)$$

(where the first two states represent the Erlang-2 distribution) gives

$$E[X] = \frac{4}{3\lambda} + \frac{p}{3\mu_1} + \frac{1-p}{3\mu_2}$$

showing that Eq. (11) does not hold in this example where $\mathbf{Q}$ is not doubly stochastic. The reason is that for irreducible doubly stochastic routing matrices $\beta\mathbf{Q} = \beta, \beta\mathbf{1}^T = 1$ holds if $\beta = \frac{1}{N}\mathbf{1}$, thus giving the possibility to establish a relation between the block moments and the moments of the whole MAP being independent of the routing matrix $\mathbf{Q}$ as expressed in Eq. (11). Concerning MAP fitting Eq. (11) is of special interest, since it suggests to first fit the distributions of the blocks. This step might also include the choice of a proper value for N. Once the block distributions are identified Eq. (12) can be employed to determine the correlation structure by selecting a suitable routing matrix.

IV. Circulant SMAPs

As shown in the last section a doubly stochastic routing matrix imposed on a block representation of a MAP gives us concise expressions for the MAP's characteristics in terms of the characteristics of the blocks. A natural question is thus how to construct doubly stochastic matrices.

Birkhoff's theorem states that every doubly stochastic matrix is a convex combination of finitely many permutation matrices [13]. Furthermore they form a semigroup under matrix multiplication. Doubly stochastic matrices have been investigated sufficiently, see e.g. [13], [17]. Trivial examples of doubly stochastic matrices are

- $\mathbf{Q} = (q(i,j))$ with $q(i,j) := \frac{1}{N}, \forall i,j \in \mathcal{N}$ and
- $\mathbf{Q} = (q(i,j))$ with $q(i,j) := \delta(((i+1) \mod N), (j \mod N)), \forall i,j \in \mathcal{N}$. This matrix describes an SMAP where the blocks are daisy-chained and after leaving block N the process enters block 1. Note that $\mathbf{Q}^k(i,j) = \delta(((i+k) \mod N), (j \mod N))$.

Both examples are special cases of so-called circulant matrices [4], [7] which have several nice properties and provide an option to define doubly stochastic matrices straightforward as follows.

Let $q_1, \ldots, q_N$ be the probabilities of a discrete probability distribution, i.e. $q_i \geq 0$ and $\sum_{i=1}^N q_i = 1$. Define a non-diagonal circulant matrix (i.e. $q_1 < 1$) by

$$\mathbf{Q} := \begin{pmatrix} q_1 & q_2 & \cdots & q_{N-1} & q_N \\ q_N & q_1 & q_2 & & q_{N-1} \\ \vdots & q_N & q_1 & \ddots & \vdots \\ q_3 & & \ddots & \ddots & q_2 \\ q_2 & q_3 & \cdots & q_N & q_1 \end{pmatrix}$$

where each row results from a cyclic shift of the row above it. By construction $\mathbf{Q}$ is a doubly stochastic and irreducible(!) matrix. Since matrix $\mathbf{Q}$ is determined by the values $q_1, \ldots, q_N$ it is common to use the short-hand notation $circ(q_1, \ldots, q_N)$ for $\mathbf{Q}$. Circulant matrices are special cases of Toeplitz matrices [7] and the powers of circulant matrices can be calculated very efficiently, since every circulant matrix $\mathbf{Q}$ has eigenvectors

$$\mathbf{v}^m = \frac{1}{\sqrt{N}} \left(1, e^{-2\pi i m/N}, \ldots, e^{-2\pi i m(N-1)/N}\right)$$

and corresponding eigenvalues

$$\psi^{(m)} = \sum_{k=1}^{N} q_k e^{-2\pi i m(k-1)/N}$$

for $m = 0, \ldots, N-1$, where i is here the complex unit and π Ludolph's number [7]. $\mathbf{Q}$ can be expressed in the form $\mathbf{Q} = \mathbf{V}\Psi\mathbf{V}^*$ where $\mathbf{V}$ has the eigenvectors $\mathbf{v}^m$ as columns and $\Psi = Diag(\psi)$ is a diagonal matrix with the eigenvalues of $\mathbf{Q}$ at the diagonal. $\mathbf{V}^*$ is the conjugate transpose of $\mathbf{V}$ for which one can show here that $\mathbf{V}$ is unitary (i.e. $\mathbf{V}^* = \mathbf{V}^{-1}$), so that $\mathbf{V}^*\mathbf{V} = \mathbf{I}$ holds giving $\mathbf{Q}^i = \mathbf{V}\Psi^i\mathbf{V}^*$. Thus the powers of $\mathbf{Q}$ can be calculated very efficiently, because the eigenvectors $\mathbf{v}^m$ do only depend on N and the eigenvalues $\psi^{(m)}$, which are the elements of the diagonal matrix Ψ, can be efficiently determined by applying fast Fourier transform algorithms.

In the following we will have a closer look on SMAPs with circulant routing matrices. If $\mathbf{Q}$ is a doubly stochastic and circulant matrix, we will call an SMAP a **circulant SMAP**.

As mentioned, [21] shows that a (nonredundant) MAP can be characterised by the moments $E[X^i], i = 1, \ldots 2n-1$ and the lag-1 joint moments $E[X_0^i X_1^j], i,j = 1, \ldots, n-1$ where n is the number of states(!). Therefore it seems to be sufficient to consider the equations

$$NE[X^i] = \mathbf{E}^i \mathbf{1}^T \quad (13)$$

$$NE[X_0^i X_1^j] = \mathbf{E}^i \mathbf{Q} \mathbf{E}^{j^T} \quad (14)$$

Surely we cannot expect to characterise the MAP with these equations, since for given N we know the number of blocks, but not the number of states of the underlying MAP. However, as mentioned before, Eqs. (13) and (14) suggest a similar approach as investigated in [14] to first fit the interarrival time distribution using Eq. (13) and then to fit the correlation structure using Eq. (14). Especially the last step is straight-forward for circulant routing matrices:

Assume that a set of distributions and thus the corresponding moments $\mathbf{E}^i$ are given all satisfying Eq. (13) and that a circulant routing matrix $\mathbf{Q} = circ(q_1, \ldots, q_N)$ has to be determined. If we define $\mathbf{q} = (q_1, \ldots, q_N)$ and $c(i,j) = NE[X_0^i X_1^j]$ Eq. (14) can be rewritten as

$$c(i,j) = \mathbf{q}\,\mathbf{U}\left(\mathbf{E}^i \otimes \mathbf{E}^j\right)^T \quad (15)$$

with

$$\mathbf{U} = (u(a,b)) \in \mathbb{R}^{N \times N^2},$$

$$u(a,b) = \delta((b-1) \text{ div } N, (b-a) \mod N),$$

$a = 1, \ldots, N, b = 1, \ldots, N^2$ and where $\otimes$ denotes the Kronecker product [3].

Example 2: Let $(N, \mathcal{B}, \mathcal{Y}, \mathbf{Q})$ be the SMAP from Ex. 1 now with unspecified routing matrix $\mathbf{Q} = circ(q_1, 1-q_1)$ and given joint moment $c(1,1)$. Since $N = 2$ we have

$$\mathbf{U} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

and thus the equation for $\mathbf{q} = (q_1, 1-q_1)$ is

$$\mathbf{q} \begin{pmatrix} \frac{4}{\lambda^2} + \left(\frac{p}{\mu_1} + \frac{(1-p)}{\mu_2}\right)^2 \\ \frac{4}{\lambda}\left(\frac{p}{\mu_1} + \frac{(1-p)}{\mu_2}\right) \end{pmatrix} = (c(1,1))$$

For the following concrete values for the parameters of the random variables Y_i: $\lambda = 1$, $\mu_1 = 1/10$, $\mu_2 = 10$, $p = 1/3$ and assuming $c(1,1) = 15$ one gets $\mathbf{q} = \left(\frac{5}{7}, \frac{2}{7}\right)$ as a solution.

If several joint moments $c(i,j)$ are given we can build a system of linear equations for the unknown vector $\mathbf{q}$ by concatenating the column vectors $\mathbf{U}(\mathbf{E}^i \otimes \mathbf{E}^j)^T$. E.g., given the joint moments $c(i,j)$ and $c(k,l)$, the system of linear equations is

$$\mathbf{q}\left(\mathbf{U}(\mathbf{E}^i \otimes \mathbf{E}^j)^T \mid \mathbf{U}(\mathbf{E}^k \otimes \mathbf{E}^l)^T\right) = (c(i,j), c(k,l))$$

with an additional equation expressing $\mathbf{q}\mathbf{1}^T = 1$.

Example 3: Let $(N, \mathcal{B}, \mathcal{Y}, \mathbf{Q})$ be an SMAP with $N = 3, \mid \mathcal{B} \mid = 3, \mathcal{Y} = \{Y_1, Y_2, Y_3\}$ where Y_1, Y_2 are from Ex. 1 and Y_3 is exponentially distributed with parameter τ, so that

$$\mathbf{E}^k = \left(\frac{(k+1)!}{\lambda^k}, p\frac{k!}{\mu_1^k} + (1-p)\frac{k!}{\mu_2^k}, \frac{k!}{\tau^k}\right)$$

With matrix

$$\mathbf{U} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

the column vectors for building the set of linear equations are given by $\mathbf{U}\left(\mathbf{E}^i \otimes \mathbf{E}^j\right)^T =$

$$\begin{pmatrix} \mathbf{E}^i(1)\mathbf{E}^j(1) + \mathbf{E}^i(2)\mathbf{E}^j(2) + \mathbf{E}^i(3)\mathbf{E}^j(3) \\ \mathbf{E}^i(1)\mathbf{E}^j(2) + \mathbf{E}^i(2)\mathbf{E}^j(3) + \mathbf{E}^i(3)\mathbf{E}^j(1) \\ \mathbf{E}^i(1)\mathbf{E}^j(3) + \mathbf{E}^i(2)\mathbf{E}^j(1) + \mathbf{E}^i(3)\mathbf{E}^j(2) \end{pmatrix}$$

If we assume the following concrete values for the parameters of the random variables Y_i: $\lambda = 1$, $\mu_1 = 1/10$, $\mu_2 = 10$, $p = 1/3$ and $\tau = 2$ and assume $c(1,1) = 15, c(1,2) = 220$ the system of linear equations is

$$\mathbf{q} \begin{pmatrix} \frac{1581}{100} & \frac{119481}{500} & 1 \\ \frac{19}{2} & \frac{6903}{50} & 1 \\ \frac{19}{2} & \frac{2737}{50} & 1 \end{pmatrix} = (15, 220, 1)$$

resulting in $\mathbf{q} = \left(\frac{550}{631}, \frac{73924}{1314373}, \frac{94799}{1314373}\right)$ as a feasible solution.

Certainly, although checking whether a circulant routing matrix exists and its determination is reduced here to the well-known problem of solving a system of linear equations, MAP fitting in general is still a non-trivial problem also for circulant SMAPs. The reason is that possible solutions of the system of linear equations depend on the chosen joint moments $c(i,j)$ and on the selected distributions of the blocks. E.g., a trivial solution to Eq. (13) is $\mathbf{E}^i = E[X^i]\mathbf{1}$ which results in an SMAP where essentially each block distribution coincides with the distribution of the SMAP. Then

$$E[X_0^i X_1^j] = \frac{1}{N} E[X^i]E[X^j]\mathbf{1}\mathbf{Q}\mathbf{1}^T = E[X^i]E[X^j]$$

indicates (as expected) that the interarrival times are independent irrespective of the choice of $\mathbf{Q}$.

V. Conclusions

As we have seen, taking advantage of structure and double stochasticity supports a different view on MAPs, where one can abstract from the level of the state space. Eq. (10)-(12) express the characteristics of a MAP on the basis of the characteristics of structural entities.

In essence an SMAP is a network of probability distributions whose entire characterisation as a probability distribution can be simply expressed by Eq. (11) due to double stochasticity. Since Eq. (11) considers the moments, feasible solutions must also imply the existence of a solution to the Hamburger (resp. Stieltjes) moment problem (cf. [1], [20]). Future research might be directed towards finding a suitable family of distributions which can be easily fitted on the basis of Eq. (11), but still span a suitable vector space for satisfying Eqs. (10) and (12). As shown in Sect. IV circulant matrices are helpful when determining a suitable routing matrix, since they are by construction doubly stochastic and what is even more important irreducible. Once the block distributions respectively their moments are given, a routing matrix can be determined by solving a system of linear equations.

Returning to the starting point of this paper we got some insight into MAPs on an abstract level. An interesting question for future research is whether circulant or at least doubly stochastic SMAPs can (approximately) represent any MAP, so that they can serve as a basis for a canonical representation. Obviously, SMAPs with general routing matrices can approximately represent any finite MAP, but as shown Eqs. (10)-(12) need not hold in that case.

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Appendix

In the following we give a detailed proof of the paper's main theorem.

Proof of Th. 2:

In the following we will calculate $E[X_0^{i_0} X_{a_1}^{i_1} \dots X_{a_k}^{i_k}] =$

$$\prod_{j=0}^{k} [i_j!] \ \alpha(-\mathbf{D_0})^{-i_0} \prod_{j=1}^{k} \left[\mathbf{P}^{(a_j - a_{j-1})} (-\mathbf{D_0})^{-i_j} \right] \mathbf{1}^T$$

by determining the individual expressions of the right side.

Lemma 1:

$$\sum_{s=0}^{j} \binom{k+s}{s} = \binom{k+j+1}{j}, \quad \forall k, j \in \mathbb{N}_0 \tag{16}$$

Proof: Proof by induction on j.

base case $j = 0$:

holds, since $\binom{k}{0} = \binom{k+1}{0} = 1$

induction step $j \to j+1$:

$$\begin{aligned} & \sum_{s=0}^{j+1} \binom{k+s}{s} \\ = & \sum_{s=0}^{j} \binom{k+s}{s} + \binom{k+j+1}{j+1} \\ = & \binom{k+j+1}{j} + \binom{k+j+1}{j+1} \\ = & \frac{(k+j+1)!}{j!(k+1)!} + \frac{(k+j+1)!}{(j+1)!k!} \\ = & (k+j+1)! * \frac{(j+1)+(k+1)}{(k+1)!(j+1)!} \\ = & \binom{k+j+2}{j+1} \end{aligned}$$

■

Theorem 3: Let $(N, \mathcal{B}, \mathcal{Y}, \mathbf{Q})$ be an SMAP with doubly stochastic matrix $\mathbf{Q}$, then we have for $k \geq 1$:

$$(-\mathbf{D_0})^{-k}((i,b,s),(j,c,t)) = \delta((i,b),(j,c))\gamma(s,t+1)\mu(i,b)^{-k} \binom{k-1+t-s}{t-s} \tag{17}$$

$$\mathbf{P}^k((i,b,s),(j,c,t)) = \delta(t,1)p(j,c)\mathbf{Q}^k(i,j) \tag{18}$$

$$\alpha(i,b,s) = \frac{1}{N}\delta(s,1)p(i,b) \tag{19}$$

Proof:

Eq. (17): by induction.

base case $k = 1$:

We show $\delta((i,b,s),(l,d,u)) = \sum_{j,c,t} \mathbf{D_0}^{-1}((i,b,s),(j,c,t))\mathbf{D_0}((j,c,t),(l,d,u))$

$$\sum_{j=1}^{N}\sum_{c=1}^{m(j)}\sum_{t=1}^{r(j,c)} \delta((i,b),(j,c))\gamma(s,t+1)\mu(i,b)^{-1}\delta((j,c),(l,d))\mu(j,c)[-\delta(t,u-1)\gamma(t,r(j,c))+\delta(t,u)]$$
$$= \delta((i,b),(l,d))\sum_{t=1}^{r(i,b)}\gamma(s,t+1)[-\delta(t,u-1)\gamma(t,r(i,b))+\delta(t,u)]$$
$$= \delta((i,b),(l,d))[-\gamma(s,u)\gamma(u-1,r(i,b))+\gamma(s,u+1)]$$
$$= \delta((i,b),(l,d))[-\gamma(s,u)+\gamma(s,u+1)]$$
$$= \delta((i,b),(l,d))\delta(s,u)$$

induction step

$$(-\mathbf{D_0})^{-(k+1)}((i,b,s),(j,c,t))$$
$$= \sum_{l=1}^{N}\sum_{d=1}^{m(l)}\sum_{u=1}^{r(l,d)} \delta((i,b),(l,d))\gamma(s,u+1)\mu(i,b)^{-1}\delta((l,d),(j,c))\gamma(u,t+1)\mu(l,d)^{-k}\binom{k-1+t-u}{t-u}$$
$$= \delta((i,b),(j,c))\mu(i,b)^{-(k+1)}\gamma(s,t+1)\sum_{u=s}^{t}\binom{k-1+t-u}{t-u}$$
$$= \delta((i,b),(j,c))\mu(i,b)^{-(k+1)}\gamma(s,t+1)\binom{k+t-s}{t-s} \quad \text{using Lemma 1}$$

Eq. (18): by induction.

base case $k=1$:
$\mathbf{P} = (-\mathbf{D_0})^{-1}\mathbf{D_1}$.

$$\mathbf{P}((i,b,s),(j,c,t))$$
$$= \sum_{l=1}^{N}\sum_{d=1}^{m(l)}\sum_{u=1}^{r(l,d)} \delta((i,b),(l,d))\gamma(s,u+1)\mu(i,b)^{-1}\delta(t,1)\delta(u,r(l,d))q(l,j)p(j,c)\mu(l,d)$$
$$= \sum_{u=1}^{r(i,b)}\gamma(s,u+1)\delta(t,1)\delta(u,r(i,b))q(i,j)p(j,c)$$
$$= \delta(t,1)p(j,c)\mathbf{Q}(i,j)$$

induction step

$$(\mathbf{P})^{(k+1)}((i,b,s),(j,c,t))$$
$$= \sum_{l=1}^{N}\sum_{d=1}^{m(l)}\sum_{u=1}^{r(l,d)} \delta(u,1)p(l,d)\mathbf{Q}(i,l)\delta(t,1)p(j,c)\mathbf{Q}^{k}(l,j)$$
$$= \delta(t,1)p(j,c)\sum_{l=1}^{N}\mathbf{Q}(i,l)\mathbf{Q}^{k}(l,j) \quad \text{since } \textstyle\sum_d p(l,d)=1, \quad \forall l\in\mathcal{N}.$$
$$= \delta(t,1)p(j,c)\mathbf{Q}^{k+1}(i,j)$$

Eq. (19): We show $\alpha\mathbf{P} = \alpha$ by

$$\begin{aligned}
&\alpha(j,c,t) \\
&\quad = \sum_{i=1}^{N}\sum_{b=1}^{m(i)}\sum_{s=1}^{r(i,b)} \frac{1}{N}\delta(s,1)p(i,b)\delta(t,1)p(j,c)q(i,j) \\
&\quad = \frac{1}{N}\delta(t,1)p(j,c)\sum_{i=1}^{N} q(i,j)\sum_{b=1}^{m(i)} p(i,b) \\
&\quad = \frac{1}{N}\delta(t,1)p(j,c)
\end{aligned}$$

Obviously $\alpha\mathbf{1}^T = 1$ holds. ■

Lemma 2: Define for $g = \{0,\dots,k-1\}, k \in \mathbb{N}$

$$\begin{aligned}
\mathbf{v}_g &:= \prod_{j=k-g}^{k} [i_j!]\left[\mathbf{P}^{(a_j-a_{j-1})}(-\mathbf{D_0})^{-i_j}\right]\mathbf{1}^T \\
\mathbf{w}_g &:= \prod_{j=k-g}^{k} \left[\mathbf{Q}^{(a_j-a_{j-1})}Diag(\mathbf{E}^{i_j})\right]\mathbf{1}^T
\end{aligned}$$

Then $\forall i \in \mathcal{N}$:

$$\mathbf{v}_g(i,b,s) = \mathbf{w}_g(i),\ \forall b \in \mathcal{M}(i), s \in \mathcal{R}(i,b)$$

Proof: by induction on g.

base case $g = 0$:

$$\begin{aligned}
&\mathbf{v}_0(i,b,s) \\
&\quad = \left([i_k!]\left[\mathbf{P}^{(a_k-a_{k-1})}(-\mathbf{D_0})^{-i_k}\right]\mathbf{1}^T\right)(i,b,s) \\
&\quad = \sum_{j=1}^{N}\sum_{c=1}^{m(j)}\sum_{t=1}^{r(j,c)}\sum_{l=1}^{N}\sum_{d=1}^{m(l)}\sum_{u=1}^{r(l,d)} [i_k!]\delta(u,1)p(l,d)\mathbf{Q}^{(a_k-a_{k-1})}(i,l) \\
&\qquad\qquad \delta((l,d),(j,c))\gamma(u,t+1)\mu(l,d)^{-i_k}\binom{i_k-1+t-u}{t-u} \\
&\quad = \sum_{j=1}^{N}\sum_{c=1}^{m(j)}\sum_{t=1}^{r(j,c)} [i_k!]p(j,c)\mathbf{Q}^{(a_k-a_{k-1})}(i,j)\mu(j,c)^{-i_k}\binom{i_k-1+t-1}{t-1} \\
&\quad = \sum_{j=1}^{N}\sum_{c=1}^{m(j)} \mathbf{Q}^{(a_k-a_{k-1})}(i,j)[i_k!]p(j,c)\mu(j,c)^{-i_k}\binom{i_k+r(j,c)-1}{r(j,c)-1} \quad \text{using Lemma 1} \\
&\quad = \sum_{j=1}^{N} \mathbf{Q}^{(a_k-a_{k-1})}(i,j)\mathbf{E}^{i_k}(j) \\
&\quad = \left(\mathbf{Q}^{(a_k-a_{k-1})}Diag(\mathbf{E}^{i_k})\mathbf{1}^T\right)(i) = \mathbf{w}_0(i)
\end{aligned}$$

induction step $g \mapsto g+1$:

$$
\begin{aligned}
&\mathbf{v}_{g+1}(i,b,s)\\
&\quad= \left([i_{k-(g+1)}!]\mathbf{P}^{(a_{k-(g+1)}-a_{k-(g+2)})}(-\mathbf{D_0})^{-i_{k-(g+1)}}\mathbf{v}_g\right)(i,b,s)\\
&\quad= \sum_{j=1}^{N}\sum_{c=1}^{m(j)}\sum_{t=1}^{r(j,c)}\sum_{l=1}^{N}\sum_{d=1}^{m(l)}\sum_{u=1}^{r(l,d)}[i_{k-(g+1)}!]\delta(u,1)p(l,d)\mathbf{Q}^{(a_{k-(g+1)}-a_{k-(g+2)})}(i,l)\\
&\qquad\qquad \delta((l,d),(j,c))\gamma(u,t+1)\mu(l,d)^{-i_{k-(g+1)}}\binom{i_{k-(g+1)}-1+t-u}{t-u}\mathbf{w}_g(j)\\
&\quad= \sum_{j=1}^{N}\mathbf{Q}^{(a_{k-(g+1)}-a_{k-(g+2)})}(i,j)\mathbf{w}_g(j)\sum_{c=1}^{m(j)}[i_{k-(g+1)}!]p(j,c)\mu(j,c)^{-i_{k-(g+1)}}\binom{i_{k-(g+1)}+r(j,c)-1}{r(j,c)-1}\\
&\quad= \sum_{j=1}^{N}\mathbf{Q}^{(a_{k-(g+1)}-a_{k-(g+2)})}(i,j)\mathbf{w}_g(j)\mathbf{E}^{i_{k-(g+1)}}(j)\\
&\quad= \left(\mathbf{Q}^{(a_{k-(g+1)}-a_{k-(g+2)})}Diag(\mathbf{E}^{i_{k-(g+1)}})\mathbf{w}_g\right)(i) = \mathbf{w}_{g+1}(i)
\end{aligned}
$$

■

Finally we show

$$[i_0!]\ \alpha(-\mathbf{D_0})^{-i_0}\mathbf{v}_{k-1} = \frac{1}{N}\mathbf{E}^{i_0}\mathbf{w}_{k-1}$$

which completes the proof of Th. 2:

$$
\begin{aligned}
&[i_0!]\ \alpha(-\mathbf{D_0})^{-i_0}\mathbf{v}_{k-1}\\
&\quad= [i_0!]\ \sum_{i=1}^{N}\sum_{b=1}^{m(i)}\sum_{s=1}^{r(i,b)}\sum_{l=1}^{N}\sum_{d=1}^{m(l)}\sum_{u=1}^{r(l,d)}\alpha(l,d,u)(-\mathbf{D_0})^{-i_0}((l,d,u),(i,b,s))\mathbf{v}_{k-1}(i,b,s)\\
&\quad= [i_0!]\ \sum_{i=1}^{N}\sum_{b=1}^{m(i)}\sum_{s=1}^{r(i,b)}\sum_{l=1}^{N}\sum_{d=1}^{m(l)}\sum_{u=1}^{r(l,d)}\frac{1}{N}\delta(u,1)p(l,d)\delta((l,d),(i,b))\gamma(u,s+1)\mu(l,d)^{-i_0}\binom{i_0-1+s-u}{s-u}\mathbf{v}_{k-1}(i,b,s)\\
&\quad= \frac{1}{N}\sum_{i=1}^{N}\sum_{b=1}^{m(i)}\sum_{s=1}^{r(i,b)}[i_0!]\ p(i,b)\mu(i,b)^{-i_0}\binom{i_0-1+s-1}{s-1}\mathbf{w}_{k-1}(i) \quad \text{using Lemma 2}\\
&\quad= \frac{1}{N}\sum_{i=1}^{N}\mathbf{E}^{i_0}(i)\mathbf{w}_{k-1}(i) \quad \text{using Lemma 1 and (3)}\\
&\quad= \frac{1}{N}\mathbf{E}^{i_0}\mathbf{w}_{k-1}
\end{aligned}
$$

■